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QUANTITATIVE ASSESSMENT OF THE ACCURACY OF CONSTITUTIVE LAWS FOR PLASTICITY WITH AN EMPHASIS ON CYCLIC DEFORMATION

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The paper addresses the plastic behavior of the aluminum alloy 5086 and 5454 in the H-32 temper under cyclic load. It reports the experimental results and analyzes them with the aim to design constitutive laws which could be used in the computational analysis. The reliability of the constitutive laws is analyzed.

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QUANTITATIVE ASSESSMENT OF THE ACCURACY OF CONSTITUTIVE LAWS FOR PLASTICITY WITH AN EMPHASIS ON CYCLIC DEFORMATION

by

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ABSTRACT

This study is concerned with the form of constitutive models, determination of parameters and agreement with experimental data. Studies of plasticity and cyclic loading have generated an extensive list of constitutive models that describe deformation under cyclic loading. The models vary from empirical formulations to viscoplastic models as well as time dependent and independent models. In most cases, these models are derived and evaluated in a one dimensional setting and are generalized to three dimensions using a multidimensional yield criterion. Often the constitutive models are evaluated using only a limited or restricted set of experimental data.

One intention of this study is to examine the application of constitutive models to constant amplitude and random amplitude loading and to compare the results with experimental stress-strain data for the same conditions. This paper addresses some basic constitutive laws used in engineering and proposes a new law which leads to a well posed mathematical problem and agrees well with experimental data. The scope of this study is restricted to time and temperature independent models. The constitutive laws selected for this study are: kinematic hardening, isotropic hardening, and a Chaboche law which have been described in the literature. In addition to these three laws, we introduce and evaluate one new law, the B-L law. The experimental database is constructed from a series of constant amplitude and random amplitude strain controlled cyclic loading experiments with different mean levels performed on specimens of 5086 and 5454 aluminum alloy in the H32 temper. The major alloying elements in this work hardening material are magnesium and manganese. This particular alloy is widely used in welded structures, pressure vessels and tube for marine structures. The nominal mechanical properties are: tensile strength 40 ksi, yield strength 30 ksi and elongation 10%. The experimental data was gathered at a constant strain rate and constant temperature. Replicate tests were conducted for each strain level such that the data base consisted of 84 experiments. Dispersion of experimental measurements is due to experimental methods and variations in material composition, processing and history. Measurements of stress and strain were made with care and sensitive instruments in order to reduce the dispersion associated with experimental methods to negligible levels compared to the dispersion associated with variation of the material properties from sample to sample. Replicate measurements were made to allow a more precise determination of the parameters and their dispersion due to material variability. Determination of the

constitutive law parameters is based on a criterion of minimization of the deviation of the predicted stress and measured stress from several cyclic load histories.

In this investigation the authors: evaluate representative constitutive models in one dimensional states of stress, identify the constitutive model parameters from an experimental database, determine the dependence of the parameters upon the material, strain history, and mean strain level by analysis of variance procedures and determine by statistic means the significance of various factors on material behavior. The agreement of stresses computed from the selected constitutive laws with the experimental data is evaluated by a relative error measure. Constitutive law parameters for error determination are taken as the average over the entire database of optimal parameters determined for each sample. The results in terms of the relative error measure indicate that: the range of variability of material response is 10% with a mean of 5%, the range for the kinematic and isotropic laws is up to 40% with a mean of 35%, and lastly the range for the Chaboche and B-L constitutive law is up 25% with a mean of 15%. Larger discrepancies are observed when parameters of the constitutive laws are estimated from material property data published in handbooks and the literature. In the inverse problem when stress is prescribed and strain is the dependent variable the error is up to 150% for an optimal set of constitutive law parameters. It can be expected that in a three dimensional setting the accuracy of the prediction of the constitutive laws will degrade from these values. Because these discrepancies are large they must be taken into account in any engineering analysis involving plasticity. Although this study is specific to two particular alloys, 5086 and 5454 aluminum, the concepts and methods are applicable to assessing accuracy of constitutive laws for other alloys, load histories and states of stress.

1. INTRODUCTION

Plastic behavior of materials, especially metals, has been in the center of interest for a considerable time. We mention the following studies and refer the reader to the many references cited therein [1-6]. The reasons are two fold: The first is to understand the mechanisms of material behavior and second to formulate the constitutive laws of plasticity in a quantitative form so that they may be used in computational analyses. The second aspect is obviously of major importance for engineering applications. Nevertheless, the first aspect could significantly help in design of the qualitative form of prospective laws. See Drucker [4] for an interesting discussion. The constitutive laws unavoidably have a phenomenological character and their reliability has to be assessed in relation to the goals of intended use. In this paper, we will address these questions of constitutive modeling in conjunction with the behavior of 5086 and 5454 aluminum alloy in the H32 temper.

Our major goal is the assessment of plasticity laws for applications in computational analysis. Today such analyses are possible for complex three-dimensional problems, due to the development of computer technology. Obviously, such analyses can be directed to problems of high precision (e.g. turbine aircraft engines). In these applications the material is under very strict quality control and experiments to determine performance and properties are tailored to the specific use of the material. A second possibility is the use of these materials for more routine applications or standard problems where the material is specified by a standard commercial mark such as 5086-H32 or 5454-H32. The material source for these applications is usually from a warehouse without as much concern with respect to the manufacturer (e.g. Alcan, Alcoa, Kaiser, Reynolds, etc.) and initial form or shape of the material (sheets, plates, etc.). As an example, let us mention elastic and plastic data for the 5454-H32 aluminum alloy from [7]. In Table 1.1 are reported some of the data for sheet (nominal thickness 0.2") and plate (nominal thickness 0.4").

Table 1.1. Material properties for the 5454-H32 aluminum alloy [7].

	0.2% Yield Strength S_Y, ksi	Ultimate Strength S_u, ksi	Elongation 1" gauge Elong %	Reduction in Area RA %	Modulus of Elasticity E, ksi	Ultimate Yield Strength Ratio S_u/S_Y
Plate	25.5	40.0	17.6	28.1	10,100	1.6
Sheet	31.1	42.6	15.7	26.6	10,400	1.4

In [7] no statistical data are reported on the variation of these properties and it is not clear how many replica measurements were performed and from what position at the sheet the sample was taken. An exhaustive search of the literature reveals that data published in the literature and in handbooks are neither accompanied with basic statistics nor are these statistical data referenced. In fact, we have not found any consistent reporting of statistical data or analysis in the literature [8]. From Table 1.1 we see that the variation of modulus of elasticity is 3% while the yield strength variation is more than 20%. This shows that material properties are very sensitive to various factors associated with processing of plate and sheet and that to consider the material in an ideal laboratory condition is not sufficient for practical applications and could be misleading. We remark that annealing procedures could change the strain hardening achieved in processing these alloys to an H32 temper. Further, we must recognize that plasticity properties in materials have memory and depend on the stress and strain history and various internal mechanisms to the material.

From a practical point of view, it is not possible to experimentally study the general three-dimensional stress/strain/time relation. We therefore restrict our study to sheet material and one-dimensional stress strain relations. Two materials of very similar composition were selected and 42 samples randomly selected from one sheet of 5086-H32 aluminum and 42 samples randomly selected from one sheet of 5454-H32 aluminum were prepared. The goal of the research is to study various questions concerning these two materials. Unfortunately, the results of experimental measurements reflect not only material behavior but also the influence of experimental instrumentation. Hence, a main problems is to distinguish between the response of the material and the influence of experimental instrumentation. This task is not easy and guaranteed separation could lead to pessimistic results. In some instances, such pessimism does not influence the basic conclusions, but for more precise analyses such pessimism could adversely influence the conclusions. We have made a maximally careful analysis to distinguish between material response and the effects of instrumentation which is based on: a) estimates of the accuracy of the measurement instrumentation; b) statistical analyses of measured data and; c) inclusion of corrections to measured data to eliminate the influence of instrumentation as much as possible (especially systematic effects). These procedures will be explained later.

We addressed two categories of questions: The first is material behavior and assessment of the effect of various factors: a) Is there any significant difference between material behavior under regular constant amplitude cyclic strain and random amplitude

cyclic strain? b) How is material influenced by mean strain level? c) How does material behavior differ for materials of very similar composition? Because of limited resources we were not able to analyze many other important factors such as effects of manufacturing or processing (of the same commercial material) and the effect of sheet thickness (refer to the sheet-plate difference mentioned in Table 1.1).

The second category of questions concerns: Validity of the constitutive laws: a) Determination of parameters and initial conditions in various constitutive laws from experimental data. b) Analysis of the validity (acceptance) of a particular law and its accuracy. c) Statistical characteristics of optimal parameter values, especially if they have physical meaning. d) Influence of uncertainties in the laws and in the results of computational analyses.

2. DESCRIPTION OF THE MATERIAL

We have studied two very similar materials, 5086-H32 and 5454-H32 aluminum alloy in sheet form (0.2" nominal thickness). The principal alloying element in the 5000 series aluminum alloys is magnesium which may range from 1 to 5 percent and is often combined with lesser amounts of manganese and or chromium. The 5000 series alloy are available in a wide variety of H tempers. The selection of these materials was influenced by the following factors: The material is generally available and used in many applications. It exhibits rate independent nonlinear plastic behavior under cyclic loading, is stable and does not undergo microstructural changes during cyclic deformations, has moderate strength, good workability and elongation to failure, and is easy to machine for the purpose of specimen fabrication.

The nominal material properties reported by the Aluminum Association [9] and the American Society for Metals [10] are slightly different. Nominal properties for the two alloys are summarized in Table 2.1.

Table 2.1 Nominal mechanical properties of 5086-H32 and 5454-H32 aluminum [9,10].

Property	5086-H32	5454-H32
Tensile elastic modulus (ksi)	10,300	10,100
Compression elastic modulus (ksi)	10,500	10,300
Poisson's ratio	0.33	0.33
Tensile strength (ksi)	40-47	36-44
Yield strength (ksi)	min 28	min 26
Elongation (%)	5-12	5-12

The tensile and compressive yield strengths of both alloys are approximately the same. The shear yield strength is about 55% of the tensile yield strength.

In Table 2.2 we report the chemical composition of six samples selected randomly from both lots of material under consideration. The detection limits for copper, lead, tin and zinc for the chemical analysis shown in Table 2.2 are 0.01%. From Table 2.2 we may conclude that: a) the chemical composition is within admissible limits; and b) the chemical composition does not vary significantly in one sheet.

The microstructures of the material used in this study are shown in Figure 2.1. The dark areas are precipitates with 5454 showing slightly coarser precipitates. The grain boundaries are not well defined. An electron microprobe analysis indicates that the precipitates contain Mg, Fe, Mn and Cr. Both materials have similar microstructure. In

Table 2.2 Chemical composition of six samples in percent by weight maximum unless shown as a range [9,10].

	5086-H32			
Sample	1	2	3	Nominal
Element				
Chronium	0.10	0.10	0.10	0.05-0.25
Copper	–	0.02	0.01	0.10
Iron	0.40	0.38	0.45	0.50
Magnesium	3.74	3.86	3.76	3.5-4.5
Manganese	0.47	0.47	0.47	0.2-0.7
Nickle	0.02	0.02	0.02	0.05
Lead	–	–	–	–
Silicon	0.08	0.08	0.08	0.040
Tin	–	–	–	–
Titanium	0.03	0.03	0.03	0.15
Zinc	–	–	–	0.25

	5454-H32			
Sample	1	2	3	Nominal
Element				
Chronium	0.09	0.09	0.09	0.05-0.20
Copper	–	–	–	0.10
Iron	0.27	0.28	0.28	0.40
Magnesium	2.62	2.56	2.52	2.4-3.0
Manganese	0.66	0.67	0.67	0.5-1.0
Nickle	0.02	0.02	0.02	0.05
Lead	–	–	–	–
Silicon	0.04	0.06	0.06	0.25
Tin	–	–	–	–
Titanium	0.02	0.02	0.02	0.20
Zinc	–	–	–	0.25

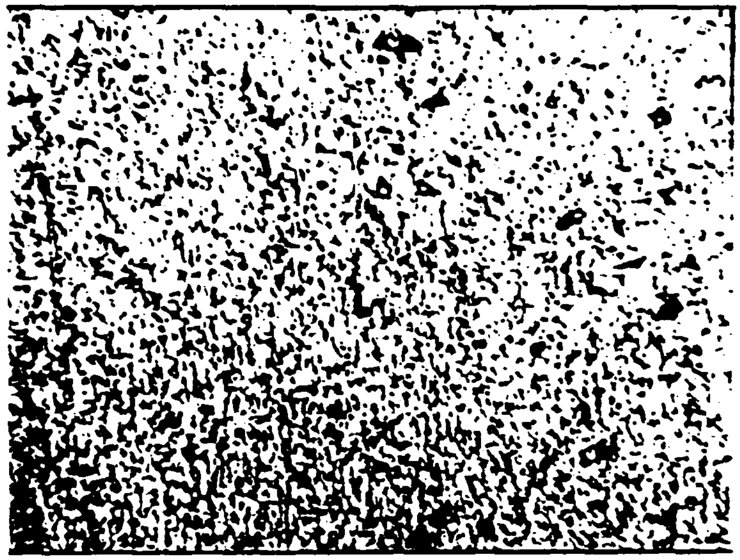

Figure 2.1. Microstructures of a) 5086 and b) 5454 aluminum. (500x)

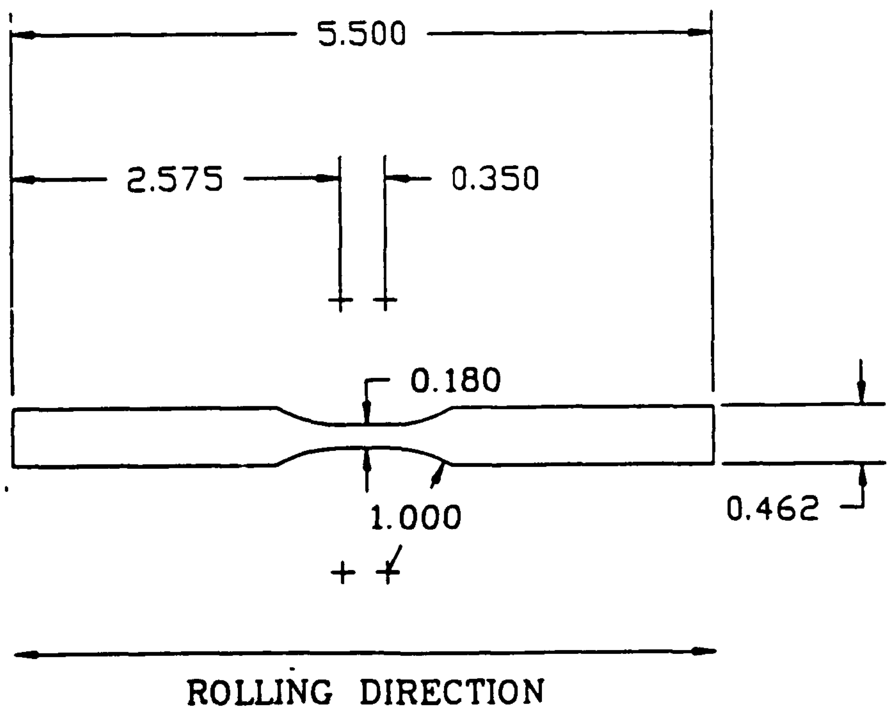

Figure 2.2. Fatigue sample geometry and dimensions.

addition, hardness measurements were taken (HRB Rockwell B) on every sample. The results for each material are: 5086: 32-40 (mean 36.2, standard deviation 2.3) and 5454: 26-35 (mean 31.1, standard deviation 2.4).

Specimens as shown in Figure 2.2 were fabricated from each sheet of material. The specimens were of rectangular cross section with a thickness of 0.2" (the nominal sheet thickness). Each lot of specimens consisted of 42 samples from each sheet. The position of the samples in the sheet was randomized and the samples were oriented with the direction of sheet fabrication (rolling direction).

3. EXPERIMENTAL INSTRUMENTATION AND EXPERIMENTAL TECHNIQUE

The detailed description of the experimental instrumentation is given in Appendix I. The cyclic plasticity experiments were performed under strain control at room temperature (22°C), relative humidity of 50%, maximum strain of $|\varepsilon| \le 1.1\%$ and a strain rate of $|\dot{\varepsilon}| = 0.001$ per second with stress $|\sigma|$ in the range 0-35 ksi. There were two types of strain histories considered: a) A cyclic constant amplitude strain in the form of a triangular function of 1000 reversals with a mean strain $\bar{\varepsilon}$ = -0.006, -0.004, -0.002, 0.000, +0.002, +0.004, +0.006 and a strain range of 0.01. Figure 3.1 illustrates the constant amplitude cyclic history for a mean $\bar{\varepsilon}$ = 0.006. b) A random amplitude strain in the form of a piecewise linear function with the same mean levels, $\bar{\varepsilon}$, and with peaks selected as random (pseudo random) numbers uniformly distributed in the interval (-0.005, 0.005). Two random sequences of peaks created by different seeds in the random number generator were used to create a history of 1000 reversals. In Figure 3.2 we show part of the random strain history for a seed of one.

Each strain history was applied to two samples to address the question of reproducibility. The continuous stress and strain data, $\sigma(t)$ and $\varepsilon(t)$ for each specimen, was sampled at discrete time intervals T. So σ and ε are observed at sampling instants $t_k = kT$, $k = 1, 2, \ldots$. In the constant amplitude history, approximately 40 data samples were taken between reversals of the strain rate. Two reversals constitute one cycle of the history. A window of data samples is then defined as a range of data samples from a beginning cycle index, b, to an ending index, e $\{w_{b,e} | \, k_b T \le t_k \le k_e T\}$. To address questions on effects of the number of cycles we considered eleven different windows in the time histories from cycle indices of 0-10, 0-20, 0-50, 0-100, 0-200, 0-500, 10-20, 20-50, 50-100, 100-200 and 200-500. The stress and strain reported is defined as engineering stress and strain without correction for geometric changes which occur during the deformation process.

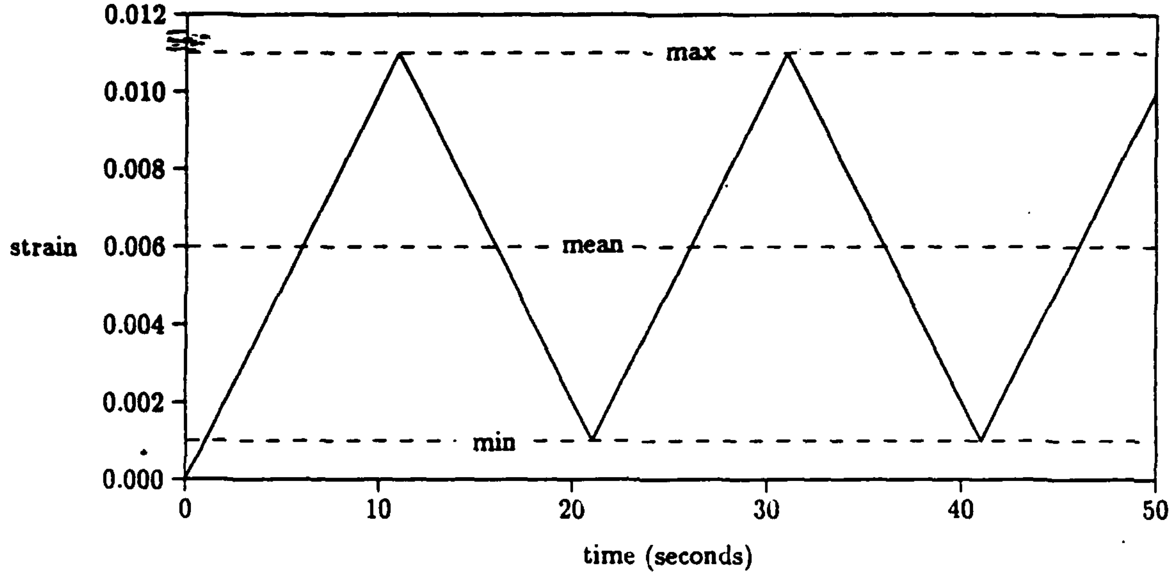

Figure 3.1. Strain, $\varepsilon(t)$, for the cyclic constant amplitude strain history.

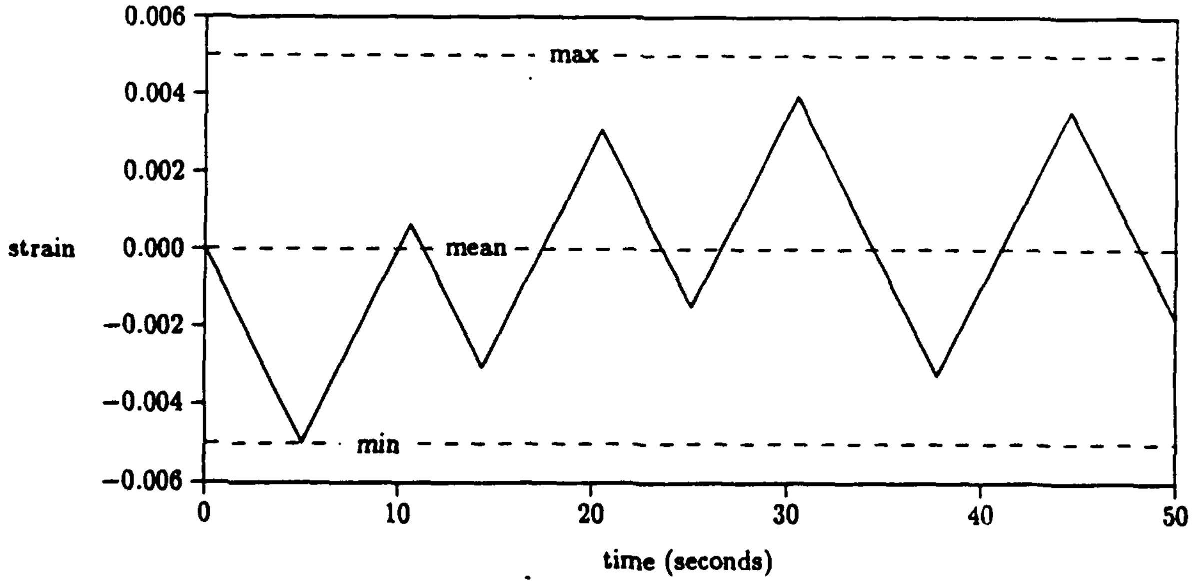

Figure 3.2. Strain, $\varepsilon(t)$, for the first cyclic random history.

4. STATISTICAL ANALYSES OF THE STRESS AND STRAIN MEASUREMENT

The major part of this paper is the analysis of material behavior under cyclic load based on the measured relation between strain and stress. This relation is obviously influenced by the measurement error associated with the instrumentation. Hence it is essential to analyze these effects as thoroughly as possible. In Appendix I estimates of the measurement errors are given by addressing tolerances in the instrumentation. In this section we will analyze these errors by various statistical means to obtain realistic estimates of the difference between the true and measured data and effect of these errors on the constitutive relations under considerations.

4.1 Analysis of the Strain Measurement

This discussion of strain accuracy is focused on the constant amplitude cyclic history. Based on instrumentation we distinguish between three values of strain: a) intended, b) measured, and c) true. Our primary interest is the difference between measured and true strain. Strain and stress are reported at points of "intended" strain which is programmed into the servohydraulic instrumentation. The reported values in this section are subsets of all measured data. Approximately 40,000 measurements of stress and strain are recorded and numbered in sequence for every specimen. Except in the close neighborhood of strain reversals (peaks), where an influence of inertia in the instrumentation is present and control accuracy is not as good, the increments or difference of strain at two adjacent sampling intervals should be constant to within the measurement error. The results of 10 representative cases out of 28 randomly selected cases are reported in Table 4.1. Note that increasing $(\dot{\varepsilon} > 0)$ and decreasing $(\dot{\varepsilon} < 0)$ strains are separately analyzed. The mean, standard deviation and maximal difference between the mean and the measurements are tabulated over the entire history of 1000 reversals. Approximately forty strain increments occur between each reversal of strain rate.

Table 4.1. Increments of strain between sampling instants.

	Increasing Strain			Decreasing Strain		
Case	Mean	Std Dev	Max	Mean	Std Dev	Max
1	0.2599D-3	0.3820D-5	0.163D-4	-0.2599D-3	0.4238D-5	0.213D-4
2	0.2600D-3	0.3757D-5	0.188D-4	-0.2600D-3	0.3690D-5	0.169D-4
3	0.2601D-3	0.3527D-5	0.182D-4	-0.2601D-3	0.3807D-5	0.180D-4
4	0.2606D-3	0.3685D-5	0.195D-4	-0.2607D-3	0.3961D-5	0.186D-4
5	0.2605D-3	0.3852D-5	0.176D-4	-0.2605D-3	0.4421D-5	0.226D-4
6	0.2600D-3	0.3782D-5	0.201D-4	-0.2600D-3	0.3667D-5	0.156D-4
7	0.2606D-3	0.3527D-5	0.201D-4	-0.2606D-3	0.3684D-5	0.245D-4
8	0.2600D-3	0.3704D-5	0.162D-4	-0.2600D-3	0.3775D-5	0.174D-4
9	0.2605D-3	0.3662D-5	0.202D-4	-0.2605D-3	0.3803D-5	0.189D-4
10	0.2600D-3	0.3767D-5	0.195D-4	-0.2600D-3	0.3715D-5	0.163D-4

Data for the other 18 cases considered are completely analogous. From Table 4.1 and additional statistical analysis we find that the error in the strain increment has a random character with a maximum of $\approx 0.2\ 10^{-4}$. In fact, we can assume a smaller error for an individual history because the increment is the difference of two measurements. This value of 20 microstrain is compared with the repeatability value based on instrumentation tolerances (see Appendix I) of 9.05 microstrain ($6.4 \times \sqrt{2}$ because of the difference in two sequential data points).

A statistical analysis of a representative subset of the experimental data is reported in Table 4.2. The differences of two time histories with the same intended strain and the differences of averages of these two histories are tabulated for five cases with different mean strain which lead to a maximal strain difference. The case titled 1-2 is the difference of the time histories for sample 1 and 2, 3-4 is similarly defined, and (1+2)/2–(3+4)/2 is the difference after averaging histories 1 and 2 and histories 3 and 4. A normality test on each data set was passed at a $p = 0.01$ level.

Table 4.2. Statistics for the difference of two strain histories with the same intended strain ($\times 10^5$).

Mean Strain		1-2	3-4	(1+2)/2 - (3+4)/2
	mean	-1.085	2.145	-0.272
	std dev	1.821	1.762	1.159
0.006	skewness	0.015	-0.072	-0.042
	range	12.56	10.87	8.125
	mean	3.614	0.973	0.340
	std dev	1.699	1.829	1.170
0.004	skewness	-0.066	-0.085	0.143
	range	10.482	11.254	11.865
	mean	4.931	1.727	-0.134
	std dev	1.637	1.877	1.336
0.000	skewness	0.002	0.026	0.041
	range	10.99	11.97	13.035
	mean	-0.139	1.436	-0.080
	std dev	1.621	1.689	1.266
-0.004	skewness	0.016	0.021	-0.089
	range	9.13	10.289	8.54
	mean	0.983	1.982	-0.309
	std dev	1.423	1.808	1.034
-0.006	skewness	-0.008	0.020	0.084
	range	8.93	11.51	7.76

As expected the standard deviation for the difference of averages is $\approx (\sqrt{2})^{-1}$ smaller than the difference of individual cases. This suggests that the cases are not correlated. The reported differences reflect not only errors in the relative measurements but also errors in offsets of individual cases. The maximal range is of the order $1\ 10^{-4}$ which is larger than $0.2\ 10^{-4}$ reported in Table 4.1. This value of 100 microstrain is compared with the worst case value based on instrumentation tolerances (see Appendix I) of 177 microstrain ($125 \times \sqrt{2}$). The difference between these two values characterizes the offset and possible time drift in machine performance during an experiment which lasts about three hours. Nevertheless, the standard deviation is still $< 0.2\ 10^{-4}$.

The effect of drift in machine performance is analyzed with the autocorrelation function for case 1-2 (a mean strain of +0.006) and 3-4 (a mean strain of 0.000). Strain reversals (peaks) are numbered sequentially by an index k and by $k - k_o$ we denote the value of the autocorrelation function $k - k_o$ reversals apart. Table 4.3 shows the autocorrelation function for these two cases and implies that effects other than pure white noise are present in the instrumentation.

Table 4.3. Autocorrelation function of strain for replicate experiments.

$k - k_o$	1-2	3-4
0	1.000	1.000
1	0.9621	0.9670
2	0.9586	0.9640
3	0.9544	0.9597
4	0.9473	0.9540
5	0.9403	0.9472
10	0.8854	0.8991
15	0.8075	0.8327
20	0.7171	0.7569
30	0.5423	0.6065
50	0.4413	0.4987
100	0.2386	0.284

Based on a statistical analysis of the data we draw the following conclusions: a) The total maximal error in strain including the effect of offsets is at most of order 1.0 10^{-4} (because of differencing and lack of correlation of the measurements) with standard deviation < 0.2 10^{-4}. b) There is a slow time drift in performance of the instrumentation. c) The measured strain is more accurate that the "intended" strain (this is important in the following stress analysis).

4.2. Analysis of the Stress Measurement

Evaluating the accuracy of the stress measurement is more complex. These measurements are influenced not only by instrumentation but also by material response. Hence the difference of the stresses for two identical strain histories is influenced by: a) accuracy of the instrumentation, b) offset in strain and stress, and c) differences in behavior of the material between individual samples. Differences in material from sample to sample will be addressed in the next section on reproducibility.

As shown in the above data the strains for two samples with the same intended strain history are different. Using the conclusions drawn in the previous section and the fact that measured strain is accurate, we can correct for differences in measured strain by interpolation. The stress of sample 1 and sample 2 can then be compared at the same strain. The interpolation scheme is modified in the neighborhood of reversals to accommodate the errors due to instrumentation inertia.

Basic statistical data are reported in Table 4.4 for replica experiments. The maximal observed stress is of order 3 to 4 10^4 psi. Only the cases with maximal mean strain

leading to maximal stress in the time history and zero mean strain are reported. The table indicates that a change of mean strain in the history does not influence the accuracy (also see Section 8).

Table 4.4. Statistics for the difference of stresses for samples with the same intended strain history (psi).

Mean Strain		1-2	3-4
0.006	mean	134.28	141.52
	std dev	1086.91	1367.93
	skewness	-0.1676	0.1073
	range	6556.6	4481.7
0.000	mean	-385.43	119.21
	std dev	1206.83	470.52
	skewness	0.2606	-0.1614
	range	5657.2	2646.3
-0.006	mean	-391.28	-232.60
	std dev	983.29	310.61
	skewness	-0.0271	-0.2452
	range	4578.5	2236.3

To obtain basic information about accuracy of stress inclusive of the offset we proceed as follows: a) Data samples 1-8 after a reversal are in a linear response region of the material. Samples one and two are not reliable because of the previously mentioned error at the strain reversal. Hence, we use the data from samples 3-8 for the following analysis. b) Increments in strain, $\Delta\varepsilon_i$, stress, $\Delta\sigma_i$, and the modulus of elasticity, E_i, are computed in the five intervals under consideration. c) The average of the five modulus values (E_o) is used to compute $\Delta\sigma_i - E_o\Delta\varepsilon_i = r_i$. The values, r_i are on the order of $|r_i| < 40$ psi and exclude the effect of any offset. d) Assume the offset has the form $\overline{\sigma}_j = \sigma_j(1+\alpha_j)$, and $\overline{\varepsilon}_j = \varepsilon_j(1+\beta_j)$ where by $\overline{\sigma}_j$ and $\overline{\varepsilon}_j$ we denote the measured data and by σ_j and ε_j the true values. The range of stress is mostly influenced by a difference in material behavior between samples. Neglecting higher order terms in α_j, β_j, we obtain $\alpha - \beta = \frac{\overline{\sigma}_i}{E_o\overline{\varepsilon}_i} - 1 = \gamma$. Results for γ are a mean of 0.016, max 0.036, and std dev 0.014 for $j = 8$. From the analysis of strain data $\varepsilon_8 \approx 2\text{x}10^{-3}$ and $\beta_8 \equiv 0.02\text{-}0.04$ which results in a pessimistic estimate for α_8 of 0.05-0.07. Because we are interested in relative error with respect to the maximal σ_{mean} of $2\sigma_8$, we obtain a maximal error in stress of 2-3.5% but more likely 1-2% (300 to 800 psi). This error is compared to the worst case value based on instrumentation tolerances (See Appendix I) of 332 psi.

In what will follow we compare samples with an identical strain history and define a relative measure of error as

$$\Theta = \frac{\max_i \left(\sigma_i^{(1)} - \sigma_i^{(2)}\right)}{\max_i \frac{1}{2}\left(\sigma_i^{(1)} + \sigma_i^{(2)}\right)} \%$$

where σ_i is the stress at the time t_i and max is taken over all t_i.

From the above results we assume that the largest instrumentation or measurement error, Θ, in stress is less than 1.5%. Also note that the error measure, Θ, is the maximal difference as opposed to the difference from the mean.

To further verify the reliability of this conclusion for the error in stress, the autocorrelation function is compared for the stress difference of two samples with the same strain history. The autocorrelation for the same cases reported in Table 4.4 are shown in Table 4.5. The periodic character of the autocorrelation function is due to differences in material behavior and not the influence of instrumentation. If the autocorrelation function was biased by the instrumentation, the results in Table 4.3 and Table 4.5 would be similar. The autocorrelation for the replicate experiments which give the smallest error, Θ, for the entire set of data is reported in Table 4.6.

Table 4.5. Stress autocorrelation function for replicate experiments.

$k-k_o$	Mean Level 0.006		Mean Level 0.000		Mean Level -0.006	
	1-2	3-4	1-2	3-4	1-2	3-4
1	+1.000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000
5	+0.9165	+0.9145	+0.9097	+0.9263	+0.9145	+0.8776
10	+0.6890	+0.6813	+0.6777	+0.6633	+0.6843	+0.6655
15	+0.3573	+0.3420	+0.3475	+0.3395	+0.3527	+0.3915
20	-0.0237	-0.0478	-0.1041	-0.0193	-0.0272	+0.1062
25	-0.3927	-0.4255	-0.335	-0.3498	-0.3975	-0.1632
30	-0.6910	-0.7315	-0.6888	-0.6066	-0.7002	-0.3902
40	-0.9067	-0.9530	-0.8990	-0.7867	-0.9242	-0.5575
60	+0.1826	+0.1702	+0.1866	+0.1701	+0.1776	+0.2239
80	+0.9589	+0.9675	+0.9598	+0.9175	+0.9656	+0.8587
100	-0.2408	-0.2652	-0.2457	-0.2380	-0.2348	-0.0990
120	-0.8863	-0.9048	-0.8521	-0.7695	-0.8778	-0.5871
140	+0.3831	+0.3802	+0.3932	+0.3554	+0.3762	+0.3361
Θ	9.42%	6.34%	7.40%	3.71%	5.47%	4.17%

Table 4.6. Stress autocorrelation function for the case with minimal error, Θ of 2.72% for a mean strain of - 0.002.

$k-k_o$	1-2
1	+1.000
5	+0.8791
10	+0.6589
15	+0.3621
20	+0.0365
25	-0.2702
30	-0.5163
40	-0.6799
60	+0.1963
80	+0.8896
100	-0.1645
120	-0.6761
140	+0.35031
Θ	2.72%

The autocorrelation function of Table 4.6 is essentially the same as in Table 4.5. This also shows that, in the case where the error is minimal, Θ = 2.72%, the periodicity is due to material differences and not instrumentation. Hence, we will conclude that the error for stress due to instrumentation is less than 1%, i.e., that the true error, Θ, is equal to the computed error ± 1% or ± 1.5% at most.

5. REPRODUCIBILITY OF MATERIAL RESPONSE

In this section we address the problem of assessing reproducibility of the stress response of two replica samples under the same strain history. Three types of strain histories with seven mean levels are considered, one cyclic and two random histories. The reproducibility of material behavior is measured by

$$\Theta(w_{b,e}) = \frac{\max\limits_{w_{b,e}} |(\sigma_1 - \sigma_2)|}{\max\limits_{w_{b,e}} \frac{1}{2} |(\sigma_1 + \sigma_2)|} \%$$

where w denotes a "window" of measurement and σ_1, σ_2 are stresses for two replicate experiments. To capture the dependence of behavior on the number of cycles (2x the number of reversals) we consider a "window" defined as the number of cycles beginning at b and ending at e as described in Section 3. The 11 windows are numbered sequentially from 1-11 for the following ranges of cycle indices : 0-10, 0-20, 0-50, 0-100, 0-200, 0-500, 10-20, 20-50, 50-100, 100-200, 200-500.

The values of the coefficient Θ for the 84 experiments are reported in Table 5.1a-c. The statistical data (average, minimum, maximum, and standard deviation) are for the case when both materials are considered as one set, and when each material is considered as a separate set. The notation in the tables is defined as follows: ave rel error: Average value over the set containing both materials, ave rel error 1: Average value over the set only containing material 1 (5086-H32) and ave rel error 2: Average value over the set only containing material 2 (5454-H32). The other labels for maximum, minimum and standard deviation have analogous meaning. We see that reproducibility characterized by the coefficient Θ is in the average of the order of 5% with a maximum of up to 10%. As discussed in Section 4, we expect at most 1-1.5% error in this data to be due to measurement error. Now let us analyze these tables by statistical methods of multivariant analysis, and the analysis of variance procedures of, Wilk's Lambda, Pillai's Trace, Hotelling-Lawley Trace hypothesis tests and Tukey and Scheffe grouping techniques. All statistical tests, comparisons and conclusions are based on the error Θ.

The following conclusions are drawn at a p-= 0.05 level: a) Dependence of material reproducibility on the window is not significantly influenced by the mean strain level and the history within one material. In window 11 (200-500 cycles) the influence of mean strain level and history is significant for both materials considered as one set.

b) Reproducibility is significantly influenced by the window for both materials considered as one set. c) The effect of history is not influenced by the window. d) The effect of mean strain level is not influenced by the window.

Let us underline that we are discussing the reproducibility of material behavior, i.e., the difference of the stress response of two samples of the same material and the same strain history. Hence, the influence of the material on the factors of mean level, history, and, window could depend on parameters other than the difference of the stresses. The main reason for variation in behavior of two samples is the state of the material (or memory) of the samples.

Table 5.1a. The coefficient Θ in percent for stress in the constant amplitude strain history.

Material	Mean	0-10	0-20	0-50	0-100	0-200	0-500
5086	0.000	4.439	4.900	5.802	6.430	6.233	7.402
5454	0.000	4.268	4.232	4.100	3.989	3.860	3.708
5086	+0.002	3.043	2.956	2.880	2.860	3.798	4.720
5454	+0.002	6.482	6.307	6.125	5.977	5.783	5.551
5086	+0.004	1.983	2.354	3.471	3.870	5.118	4.907
5454	+0.004	6.112	5.941	5.748	5.600	5.528	5.790
5086	+0.006	10.476	10.210	9.973	9.814	9.644	9.418
5454	+0.006	6.488	6.382	6.415	6.723	6.709	6.434
5086	-0.002	3.171	3.065	2.945	2.855	2.845	2.722
5454	-0.002	4.344	4.219	4.075	4.308	4.570	4.469
5086	-0.004	7.280	7.047	6.787	6.610	6.482	7.356
5454	-0.004	4.685	4.594	4.486	4.370	4.234	4.076
5086	-0.006	6.345	6.168	5.950	5.802	5.596	5.475
5454	-0.006	4.356	4.284	4.193	4.467	4.338	4.174
ave rel error		5.248	5.190	5.211	5.263	5.338	5.443
ave rel error 1		5.248	5.243	5.401	5.463	5.674	6.000
ave rel error 2		5.248	5.137	5.020	5.062	5.003	4.886
min error		1.983	2.354	2.880	2.855	2.845	2.722
min error 1		1.983	2.354	2.880	2.855	2.845	2.722
min error 2		4.268	4.219	4.075	3.989	3.860	3.708
max error		10.476	10.210	9.973	9.814	9.644	9.418
max error 1		10.476	10.210	9.973	9.814	9.644	9.418
max error 2		6.488	6.382	6.415	6.723	6.709	6.434
std dev		2.068	1.955	1.819	1.781	1.614	1.692
std dev 1		2.756	2.597	2.372	2.312	2.020	2.047
std dev 2		0.978	0.945	0.957	0.959	0.950	0.955

Material	Mean	0-10	10-20	20-50	50-100	100-200	200-500
5086	0.000	4.439	4.900	5.802	6.430	6.166	7.402
5454	0.000	4.268	4.232	3.519	2.943	2.689	2.759
5086	+0.002	3.043	2.219	2.691	2.860	3.798	4.720
5454	+0.002	6.482	4.132	4.154	4.315	4.561	4.086
5086	+0.004	1.983	2.354	3.471	3.870	5.118	4.635
5454	+0.004	6.112	5.941	5.558	5.387	5.528	5.790
5086	+0.006	10.476	9.665	9.556	9.392	8.773	7.863
5454	+0.006	6.488	6.004	6.415	6.723	6.709	5.972
5086	-0.002	3.171	2.714	2.890	2.790	2.845	2.460
5454	-0.002	4.344	3.280	3.385	4.308	4.570	4.469
5086	-0.004	7.280	5.441	5.816	6.166	6.482	7.356
5454	-0.004	4.685	4.533	4.163	3.724	3.017	2.094
5086	-0.006	6.345	3.964	4.618	5.224	5.570	5.475
5454	-0.006	4.356	3.461	3.348	4.467	3.268	3.853
ave rel error		5.248	4.489	4.670	4.900	4.935	4.924
ave rel error 1		5.248	4.465	4.978	5.247	5.536	5.702
ave rel error 2		5.248	4.512	4.363	4.552	4.335	4.146
min error		1.983	2.219	2.691	2.790	2.689	2.094
min error 1		1.983	2.219	2.691	2.790	2.845	2.460
min error 2		4.268	3.280	3.348	2.943	2.689	2.094
max error		10.476	9.665	9.556	9.392	8.773	7.863
max error 1		10.476	9.665	9.556	9.392	8.773	7.863
max error 2		6.488	6.004	6.415	6.723	6.709	5.972
std dev		2.068	1.854	1.774	1.764	1.691	1.769
std dev 1		2.756	2.420	2.215	2.173	1.781	1.811
std dev 2		0.978	1.008	1.097	1.122	1.350	1.329

Table 5.1b The coefficient Θ in percent for stress in first random strain history.

Material	Mean	0-10	0-20	0-50	0-100	0-200	0-500
086	0.000	4.206	4.363	4.710	5.209	5.030	4.855
5454	0.000	2.864	2.790	3.349	3.822	3.685	3.523
5086	+0.002	5.834	5.659	6.538	6.900	7.147	6.922
5454	+0.002	4.074	3.961	3.825	3.771	3.630	3.472
5086	+0.004	1.605	1.838	3.232	4.050	4.458	4.651
5454	+0.004	4.718	4.619	4.493	4.442	4.288	4.102
5086	+0.006	6.575	6.425	6.233	6.129	7.547	7.664
5454	+0.006	6.798	6.688	6.583	6.512	6.385	6.223
5086	-0.002	4.448	5.230	5.000	4.891	4.690	4.507
5454	-0.002	4.283	4.183	4.067	4.015	3.901	4.981
5086	-0.004	1.901	1.852	2.154	3.002	3.577	3.581
5454	-0.004	4.134	4.073	4.774	5.644	5.893	6.219
5086	-0.006	4.753	4.687	4.772	5.147	5.441	5.538
5454	-0.006	4.336	4.274	4.193	4.150	4.066	3.935
ave rel error		4.324	4.332	4.566	4.835	4.981	5.012
ave rel error 1		4.189	4.293	4.663	5.047	5.413	5.388
ave rel error 2		4.458	4.370	4.469	4.622	4.550	4.636
min error		1.605	1.838	2.154	3.002	3.577	3.472
min error 1		1.605	1.838	2.154	3.002	3.577	3.581
min error 2		2.864	2.790	3.349	3.771	3.630	3.472
max error		6.798	6.688	6.583	6.900	7.547	7.664
max error 1		6.575	6.425	6.538	6.900	7.547	7.664
max error 2		6.798	6.688	6.583	6.512	6.385	6.223
std dev		1.447	1.406	1.226	1.104	1.271	1.279
std dev 1		1.719	1.666	1.437	1.186	1.337	1.332
std dev 2		1.094	1.084	0.961	0.971	1.034	1.103

Material	Mean	0-10	10-20	20-50	50-100	100-200	200-500
5086	0.000	4.206	4.363	4.710	5.209	4.505	4.829
5454	0.000	2.864	2.681	3.349	3.822	3.251	3.335
5086	+0.002	5.834	5.229	6.538	6.900	7.147	6.743
5454	+0.002	4.074	2.560	2.892	2.974	3.343	3.324
5086	+0.004	1.605	1.838	3.232	4.050	4.458	4.651
5454	+0.004	4.718	2.979	2.952	2.616	3.027	3.122
5086	+0.006	6.575	6.127	6.117	5.765	7.547	7.664
5454	+0.006	6.798	3.992	4.566	4.417	4.382	4.433
5086	-0.002	4.448	5.230	4.273	4.162	3.870	4.155
5454	-0.002	4.283	3.670	3.715	3.228	3.046	4.981
5086	-0.004	1.901	1.678	2.154	3.002	3.577	3.581
5454	-0.004	4.134	4.073	4.774	5.644	5.893	6.219
5086	-0.006	4.753	4.687	4.772	5.147	5.441	5.538
5454	-0.006	4.336	2.550	2.431	2.621	3.004	2.860
ave rel error		4.324	3.690	4.034	4.254	4.464	4.674
ave rel error 1		4.189	4.165	4.542	4.891	5.221	5.309
ave rel error 2		4.458	3.215	3.526	3.617	3.707	4.039
min error		1.605	1.678	2.154	2.616	3.004	2.860
min error 1		1.605	1.678	2.154	3.002	3.577	3.581
min error 2		2.864	2.550	2.431	2.616	3.004	2.860
max error		6.798	6.127	6.538	6.900	7.547	7.664
max error 1		6.575	6.127	6.538	6.900	7.547	7.664
max error 2		6.798	4.073	4.774	5.644	5.893	6.219
std dev		1.447	1.308	1.262	1.276	1.459	1.397
std dev 1		1.719	1.605	1.416	1.182	1.454	1.344
std dev 2		1.094	0.628	0.814	1.025	0.998	1.136

Table 5.1c The coefficient Θ in percent for stress in the second random strain history.

Material	Mean	0-10	0-20	0-50	0-100	0-200	0-500
5086	0.000	2.800	2.800	3.003	2.913	2.889	11.013
5454	0.000	4.879	4.724	5.234	5.961	5.965	6.341
5086	+0.002	3.780	3.780	4.396	5.918	5.761	6.696
5454	+0.002	5.869	5.869	5.555	5.419	5.230	4.978
5086	+0.004	6.182	6.158	5.746	5.568	5.433	5.277
5454	+0.004	3.022	3.022	2.885	3.811	3.728	3.646
5086	+0.006	1.753	2.147	2.679	2.618	2.564	2.506
5454	+0.006	5.926	6.233	5.938	5.839	5.725	5.524
5086	-0.002	2.814	3.364	3.825	4.684	5.389	5.593
5454	-0.002	6.866	6.711	6.508	6.380	6.162	5.902
5086	-0.004	5.069	5.622	5.907	6.195	5.981	5.684
5454	-0.004	6.103	5.987	5.828	5.722	5.529	5.291
5086	-0.006	2.841	2.780	2.972	3.802	5.191	5.270
5454	-0.006	5.172	5.414	5.473	5.709	5.562	5.455
ave rel error		4.505	4.615	4.711	5.039	5.079	5.655
ave rel error 1		3.606	3.807	4.075	4.528	4.744	6.006
ave rel error 2		5.405	5.423	5.346	5.549	5.414	5.305
min error		1.753	2.147	2.679	2.618	2.564	2.506
min error 1		1.753	2.147	2.679	2.618	2.564	2.506
min error 2		3.022	3.022	2.885	3.811	3.728	3.646
max error		6.866	6.711	6.508	6.380	6.162	11.013
max error 1		6.182	6.158	5.907	6.195	5.981	11.013
max error 2		6.866	6.711	6.508	6.380	6.162	6.341
std dev		1.571	1.513	1.319	1.204	1.111	1.797
std dev 1		1.419	1.406	1.232	1.342	1.301	2.365
std dev 2		1.141	1.139	1.073	0.759	0.744	0.789

Material	Mean	0-10	10-20	20-50	50-100	100-200	200-500
5086	0.000	2.800	2.756	3.003	2.807	2.889	11.013
5454	0.000	4.879	4.465	5.234	5.961	5.965	6.341
5086	+0.002	3.780	3.512	4.396	5.918	4.992	6.696
5454	+0.002	5.869	4.251	4.092	4.102	3.907	4.239
5086	+0.004	6.182	4.877	4.344	3.952	3.877	3.339
5454	+0.004	3.022	2.056	1.728	3.811	2.984	3.646
5086	+0.006	1.753	2.147	2.679	2.408	1.911	2.124
5454	+0.006	5.926	6.263	4.311	3.293	3.937	4.129
5086	-0.002	2.814	3.364	3.825	4.684	5.389	5.593
5454	-0.002	6.866	4.715	4.803	5.112	5.605	5.575
5086	-0.004	5.069	5.622	5.907	6.195	5.968	5.684
5454	-0.004	6.103	3.931	4.031	3.934	3.548	3.432
5086	-0.006	2.841	2.319	2.972	3.802	5.191	5.270
5454	-0.006	5.172	5.414	5.473	5.709	5.426	5.455
ave rel error		4.505	3.978	4.057	4.406	4.399	5.181
ave rel error 1		3.606	3.514	3.875	4.252	4.317	5.674
ave rel error 2		5.405	4.442	4.239	4.560	4.482	4.688
min error		1.753	2.056	1.728	2.408	1.911	2.124
min error 1		1.753	2.147	2.679	2.408	1.911	2.124
min error 2		3.022	2.056	1.728	3.293	2.984	3.432
max error		6.866	6.263	5.907	6.195	5.968	11.013
max error 1		6.182	5.622	5.907	6.195	5.968	11.013
max error 2		6.866	6.263	5.473	5.961	5.965	6.341
std dev		1.571	1.297	1.112	1.171	1.234	2.043
std dev 1		1.419	1.207	1.044	1.337	1.369	2.612
std dev 2		1.141	1.215	1.147	0.953	1.075	1.020

6. FORMULATION OF THE CONSTITUTIVE LAW

There are many constitutive laws proposed in the literature. Usually they are founded on physical and/or engineering interpretations. Essentially all of these laws have an analytical form with a number of parameters which have to be determined from experiments. For comprehensive reviews of this subject we refer the reader to [1,3,5,6] and references cited in these reviews. We consider a family of constitutive laws involving internal state variables based on two basic assumptions: 1) Existence of a convex yield surface and 2) The normality condition: the plastic strain increment during plastic flow is proportional to the outward normal to the yield surface. We choose this family of constitutive law for the following two reasons. First they are a generalization of some of the more commonly used engineering formulations. Secondly, when they are used in formulation of a plasticity problem they lead to a well posed mathematical problem. We only elaborate on one dimensional formulations. Although the approach is valid with proper changes to more general formulations.

For one-dimensional problems a yield surface can be described by stress σ and a set of internal (hardening) variables $\alpha = (\alpha_1, \cdots, \alpha_m),^T \in U \subset R^m$ where U is a convex set in R^m. The elastic set, $\mathcal{E}$, and the plastic set $\mathcal{P}$ are assumed to be convex and we can think of the yield function as gauge function of the $\mathcal{P}$ set. More precisely, we assume that there exists a function $F: R \times U \to R$ such that: 1) $F(\sigma,\alpha)$ is a convex function of σ,α and is piecewise analytical. 2) $F(0,0) = 0$ and 3) There are constants γ and $\Gamma > 0$ such that $0 < \gamma < \Gamma < \infty$ and $\gamma < \left| \frac{\partial F}{\partial \sigma} \right|, \left| \frac{\partial F}{\partial \alpha} \right| < \Gamma$ uniformly on the set $\{(\sigma,\alpha) \mid F(\sigma,\alpha) = Z_o\}$ for some $Z_o > 0$. We denoted

$$\frac{\partial F}{\partial \alpha} = \left(\frac{\partial F}{\partial \alpha_1}, \dots, \frac{\partial F}{\partial \alpha_m} \right)^T$$

Assuming that in the linear region the modulus of elasticity, E, is independent of α and σ, the constitutive law then reads as follows:

$$\begin{cases} \dot{\sigma} = E\dot{\varepsilon} \\ \dot{\alpha} = 0 \end{cases} \quad \text{for } (\sigma,\alpha) \in \mathcal{E}$$

$$\left.\begin{aligned}\dot{\sigma} &= \left[E - \frac{E^2\left(\dfrac{\partial F}{\partial \sigma}\right)^2}{\left(\dfrac{\partial F}{\partial \alpha}\right)^T \dfrac{\partial F}{\partial \alpha} + E\left(\dfrac{\partial F}{\partial \sigma}\right)^2}\right]\dot{\varepsilon} \\ \dot{\alpha} &= -\frac{\dfrac{\partial F}{\partial \sigma}\dot{\sigma}}{\left(\dfrac{\partial F}{\partial \alpha}\right)^T \dfrac{\partial F}{\partial \alpha}}\,\frac{\partial F}{\partial \alpha}\end{aligned}\right\} \text{for}(\sigma,\alpha) \in \mathcal{P}$$

where

$$\mathcal{E} = \left\{(\sigma,\alpha) \,|\, F(\sigma,\alpha) < Z_o \text{ or } F(\sigma,\alpha) = Z_o \text{ and } \frac{\partial F}{\partial \sigma}\dot{\sigma} \le 0\right\}$$

$$\mathcal{P} = \left\{(\sigma,\alpha) \,\middle|\, F(\sigma,\alpha) = Z_o \text{ and } \frac{\partial F}{\partial \sigma}\dot{\sigma} > 0\right\}$$

The constitutive law is in the form of a differential equation and of course depends on the initial value of the internal variable $\alpha(t)$. This constant, $\alpha(0)$, is considered an initial value parameter. As was said, the constitutive law should lead to a well posed mathematical problem. Although for various engineering laws it is not known whether or not they yield a well posed mathematical problem. The constitutive law is given by the function F, its analytical form, and involved constants. Let us discuss some laws which will be analyzed in detail.

6.1 Bilinear Pure Kinematic Law.

This law is often used in engineering applications. We write it in a form which belongs to our family. This form is not exactly the same as the form used in engineering literature, but it is equivalent. Transforming the hardening variable by an appropriate scaling the law has the following form:

$$\begin{cases}\dot{\sigma} = E\dot{\varepsilon} \\ \dot{\alpha} = 0\end{cases} \quad \text{for } (\sigma,\alpha) \in \mathcal{E}$$

$$\begin{cases} \dot{\sigma} = E_p \dot{\varepsilon} \\ \dot{\alpha} = E_p \sqrt{\dfrac{E - E_p}{EE_p}}\, \dot{\varepsilon} \end{cases} \qquad \text{for } (\sigma, \alpha) \in \mathcal{P}$$

where

$$\mathcal{E} = \left\{ (\sigma,\alpha) \;\middle|\; \begin{array}{l} F(\sigma,\alpha) = \left| \sigma - \sqrt{\dfrac{EE_p}{E-E_p}}\alpha \right| < \sigma_y \quad \text{or} \\ F_1(\sigma,\alpha) = \sigma - \sqrt{\dfrac{EE_p}{E-E_p}}\alpha = \sigma_y \quad \text{and} \quad \dot{\sigma} \le 0 \quad \text{or} \\ F_2(\sigma,\alpha) = -\sigma + \sqrt{\dfrac{EE_p}{E-E_p}}\alpha = \sigma_y \quad \text{and} \quad \dot{\sigma} \ge 0 \end{array} \right\}$$

$$\mathcal{P} = \left\{ (\sigma,\alpha) \;\middle|\; \begin{array}{l} F_1(\sigma,\alpha) = \sigma - \sqrt{\dfrac{EE_p}{E-E_p}}\alpha = \sigma_y \quad \text{and} \quad \dot{\sigma} > 0 \quad \text{or} \\ F_2(\sigma,\alpha) = -\sigma + \sqrt{\dfrac{EE_p}{E-E_p}}\alpha = \sigma_y \quad \text{and} \quad \dot{\sigma} < 0 \end{array} \right\}$$

This law has one internal variable, $\alpha(t)$,and depends on three parameters E, E_p and σ_y, (the modulus of elasticity, modulus of plasticity, and yield stress) and on one initial value parameter, the initial value of the internal variable at time $t = 0$, $\alpha(0)$. As standard initial conditions we use $\alpha(0) = 0$.

6.2. Bilinear Pure Isotropic Hardening Law

$$\begin{cases} \dot{\sigma} = E\dot{\varepsilon} \\ \dot{\alpha} = 0 \end{cases} \quad \text{for } (\sigma,\alpha) \in \mathcal{E}$$

$$\begin{cases} \dot{\sigma} = E_p \dot{\varepsilon} \\ \dot{\alpha} = E_p \sqrt{\dfrac{E - E_p}{EE_p}}\, \dot{\varepsilon} \end{cases} \qquad \text{for } (\sigma, \alpha) \in \mathcal{P}_+$$

$$
\begin{cases} \dot{\sigma} = E_p \dot{\varepsilon} \\ \dot{\alpha} = -E_p \sqrt{\dfrac{E - E_p}{E E_p}}\, \dot{\varepsilon} \end{cases} \qquad \text{for } (\sigma, \alpha) \in \mathcal{P}_-
$$

where

$$
\mathcal{E} = \left\{ (\sigma, \alpha) \left| \begin{array}{l} F(\sigma,\alpha) = \left| \sigma - \sqrt{\dfrac{E E_p}{E - E_p}}\, \alpha \right| < \sigma_y \ \text{ or} \\ \hat{F}_1(\sigma,\alpha) = \sigma - \sqrt{\dfrac{E E_p}{E - E_p}}\, \alpha = \sigma_y \ \text{ and } \quad \dot{\sigma} \le 0 \quad \text{or} \\ \hat{F}_2(\sigma,\alpha) = -\sigma + \sqrt{\dfrac{E E_p}{E - E_p}}\, \alpha = \sigma_y \ \text{ and } \quad \dot{\sigma} \ge 0 \end{array} \right. \right\}
$$

$$
\mathcal{P}_+ = \left\{ (\sigma,\alpha) \,|\, \hat{F}_1(\sigma,\alpha) = \sigma - \sqrt{\frac{E E_p}{E - E_p}}\alpha = \sigma_y \ \text{ and } \quad \dot{\sigma} > 0 \right\}
$$

$$
\mathcal{P}_- = \left\{ (\sigma,\alpha) \,|\, \hat{F}_2(\sigma,\alpha) = -\sigma - \sqrt{\frac{E E_p}{E - E_p}}\alpha = \sigma_y \ \text{ and } \quad \dot{\sigma} < 0 \right\}
$$

This is the usual isotropic hardening law written in our admissible form. It has one internal variable, $\alpha(t)$, and depends, as in the case of the kinematic law, on parameters E, E_p and σ_y and on one initial value parameter, the initial value of the internal variable at time $t = 0$, $\alpha(0)$. As standard initial condition we use $\alpha(0) = 0$.

6.3. Chaboche Law

This law belongs to a family of laws [11-13] of the smallest number of internal variables necessary for a reasonable representation of material response. It can be formulated in the differential equation form shown below, but it cannot be transformed in the form mentioned earlier which guarantees a well posed mathematical problem. The law reads

$$
\begin{cases}
\dot{\sigma} = E\dot{\varepsilon} \\
\dot{\chi} = 0 \\
\dot{R} = 0 \\
\dot{\varepsilon}_h = 0 \\
\dot{\varepsilon}_\ell = 0
\end{cases}
\text{ for }
\begin{cases}
\varepsilon_\ell < \varepsilon < \varepsilon_h \\
\varepsilon = \varepsilon_h \text{ and } \quad \dot{\varepsilon} \le 0 \text{ or} \\
\varepsilon = \varepsilon_\ell \text{ and } \dot{\varepsilon} \ge 0 .
\end{cases}
$$

$$
\begin{cases}
\dot{\sigma} = \dfrac{E\left[C(a-\chi)+b(Q-R)\right]}{C(a-\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{\chi} = \dfrac{EC(a-\chi)}{C(a-\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{R} = \dfrac{Eb(Q-R)}{C(a-\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{\varepsilon}_h = \dot{\varepsilon} \\
\dot{\varepsilon}_\ell = \dot{\varepsilon} - 2\dfrac{R}{E}
\end{cases}
\text{ for }
\begin{cases}
\varepsilon = \varepsilon_h \\
\text{and} \\
\dot{\varepsilon} > 0
\end{cases}
$$

$$
\begin{cases}
\dot{\sigma} = \dfrac{E\left[C(a-\chi)+b(Q-R)\right]}{C(a+\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{\chi} = \dfrac{EC(a-\chi)}{C(a+\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{R} = \dfrac{-Eb(Q-R)}{C(a+\chi)+b(Q-R)+E}\dot{\varepsilon} \\
\dot{\varepsilon}_h = \dot{\varepsilon} + 2\dfrac{R}{E} \\
\dot{\varepsilon}_\ell = \dot{\varepsilon}
\end{cases}
\text{ for }
\begin{cases}
\varepsilon = \varepsilon_\ell \\
\text{and} \\
\dot{\varepsilon} < 0
\end{cases}
$$

This law has four internal variables, $\chi(t)$, $R(t)$, $\varepsilon_h(t)$, and $\varepsilon_\ell(t)$ and depends on six parameters, which have the following physical interpretations: a: kinematic coefficient, b: isotropic exponent, C: kinematic exponent, Q: isotropic coefficient, ε_y: yield strain, E: modulus of elasticity, and on four initial value parameters. These are the initial values of the internal variables at time $t = 0$, $\chi(0)$, $R(0)$, $\varepsilon_h(0)$ and $\varepsilon_\ell(0)$. The standard initial conditions are $\chi(0) = 0$, $R(0) = 0$, $\varepsilon_h(0) = \varepsilon_y$ and $\varepsilon_\ell(0) = -\varepsilon_y$. As was said above this law does not belong to the admissible family previously formulated.

6.4. The B-L Law

We have designed a new law similar to the Chaboche law but belonging to the desired admissible family [14,15].

$$\left\{\begin{aligned} &\dot{\sigma} = E\dot{\varepsilon} \\ &\dot{\alpha} = 0 \\ &\dot{\beta} = 0 \end{aligned}\right\} \quad \text{for } (\sigma,\alpha,\beta) \in \mathcal{E}$$

$$\left\{\begin{aligned} &\dot{\sigma} = \frac{E\left[\left(\frac{C}{2}\alpha-\sqrt{aC}\right)+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2\right]}{\left(\frac{C}{2}\alpha-\sqrt{aC}\right)+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \\ &\dot{\alpha} = \frac{-E\left(\frac{C}{2}\alpha-\sqrt{aC}\right)}{\left(\frac{C}{2}\alpha-\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \\ &\dot{\beta} = \frac{-E\left(\frac{b}{2}\beta-\sqrt{bQ}\right)}{\left(\frac{C}{2}\alpha-\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \end{aligned}\right\} \quad \text{for } (\sigma,\alpha,\beta) \in \mathcal{P}_+$$

$$\left\{\begin{aligned} &\dot{\sigma} = \frac{E\left[\left(\frac{C}{2}\alpha+\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2\right]}{\left(\frac{C}{2}\alpha+\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \\ &\dot{\alpha} = \frac{E\left(\frac{C}{2}\alpha+\sqrt{aC}\right)}{\left(\frac{C}{2}\alpha+\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \\ &\dot{\beta} = \frac{E\left(\frac{b}{2}\beta-\sqrt{bQ}\right)}{\left(\frac{C}{2}\alpha+\sqrt{aC}\right)^2+\left(\frac{b}{2}\beta-\sqrt{bQ}\right)^2+E}\,\dot{\varepsilon} \end{aligned}\right\} \quad \text{for } (\sigma,\alpha,\beta) \in \mathcal{P}_-$$

where

$$\mathcal{E} = \left\{ (\sigma,\alpha,\beta) \;\middle|\; \begin{array}{lll} F(\sigma,\alpha,\beta) < Z_o & \text{or} & \\ F_1(\sigma,\alpha,\beta) = Z_o & \text{and} & \dot{\varepsilon} \le 0 \quad \text{or} \\ F_2(\sigma,\alpha,\beta) = Z_o & \text{and} & \dot{\varepsilon} \ge 0 \end{array} \right.$$

$$\mathcal{P}_+ = \left\{ (\sigma,\alpha,\beta) \,\middle|\, F_1(\sigma,\alpha,\beta) = Z_o \quad \text{and} \quad \dot{\varepsilon} > 0 \right\}$$

$$\mathcal{P}_- = \left\{ (\sigma,\alpha,\beta) \,\middle|\, F_2(\sigma,\alpha,\beta) = Z_o \quad \text{and} \quad \dot{\varepsilon} < 0 \right\}$$

$$F_1(\sigma,\alpha,\beta) = \frac{C}{4}\alpha^2 - \sqrt{aC}\,\alpha + \frac{b}{4}\beta^2 - \sqrt{bQ}\,\beta - \sigma$$

$$F_2(\sigma,\alpha,\beta) = \frac{C}{4}\alpha^2 - \sqrt{aC}\,\alpha + \frac{b}{4}\beta^2 - \sqrt{bQ}\,\beta - \sigma$$

$$F(\sigma,\alpha,\beta) = \max\left[F_1(\sigma,\alpha,\beta), F_2(\sigma,\alpha,\beta)\right]$$

and $Z_o = E\varepsilon_y$.

This law has two internal variables, $\alpha(t)$ and $\beta(t)$, depends on six parameters a, b, C, Q, E and ε_y, and on two initial value parameters, the initial value of the internal variables at time $t = 0$, $\alpha(0)$, $\beta(0)$. As standard initial conditions we use $\alpha(0) = 0$ and $\beta(0) = 0$.

The standard initial conditions for all four constitutive laws are based on the assumption that there is no initial history or memory in the material. This assumption is of course problematic because in our case the material in the as received condition is in the H-32 temper , i.e., it is strain hardened. These aspects will be discussed in Section 9. Nevertheless, because the initial conditions of the internal variables have different characteristics than the parameter values, we will use the standard initial condition assumption except in Section 9 where the initial conditions are treated as parameters to be determined from the material response.

7. PARAMETER ESTIMATION AND THE RELIABILITY OF CONSTITUTIVE LAWS

In this section parameters of the constitutive laws are determined. Unfortunately, in the literature there are no experimental data suitable for parameter estimation, and the relevant parameters can only be determined very crudely from basic available data and educated guesses. The determination of initial conditions for the internal variables is still more problematic. We utilize experimental data for estimation of parameter values and use these optimal parameters to assess the reliability of each constitutive law.

The procedure for estimation of these parameters is not unique, but it should depend on the goal of subsequent computation. Because the plasticity law depends heavily on the value of stress we used the following procedure. Assume that the functions $\sigma(t)$ and $\varepsilon(t)$ are known from an experiment on one sample. The strain, $\varepsilon(t)$, is the independent variable and it is either cyclic or random as explained in Section 3. Given the particular law, we can compute for every vector κ, a vector of parameter values, the predicted stress function $\sigma_{pred} = \Psi(\varepsilon(t),\kappa)$, which depends on the values of the parameters in vector κ. Now for a "window," $w_{b,e}$, the error measure is computed as

$$\Theta(w_{b,e},S,\kappa) = \frac{\max\limits_{w_{b,e}} \left| \sigma_s(w_{b,e}) - \Psi(\varepsilon(w_{b,e}),\kappa) \right|}{\max\limits_{w_{b,e}} \frac{1}{2} \left| \sigma_s(w_{b,e}) + \Psi(\varepsilon(w_{b,e}),\kappa) \right|}$$

where by S we denoted the sample under consideration. Then the vector of the parameters, κ_o, is determined so that $\Theta(w_{b,e}\ S,\kappa)$ is minimal. This choice of parameters is then referred to as optimal.

The computation of the parameters, κ, which minimize Θ is of L_∞ norm and is computationally expensive. A robust algorithm was used [15] to find the optimum. There are many local minima so that the result of the algorithm depended on an initial estimate of the parameters. Hence, the initial estimate which leads to the global minimum must also be analyzed. The computational cost, especially for the Chaboche and B-L laws, is very high because of the large size of the parameter vector κ. This vector can include either the parameters alone with standard initial conditions or the paramefers and initial value of the internal variables considered as a parameters (see Section 9).

Because a statistically significant difference was not observed between the two random strain histories, we consider only two histories in this section, the cyclic constant amplitude history and one random history. By the above mentioned procedure of minimizations, the optimal values of the parameters are determined for every sample. Because an influence of the window is observed on conclusions regarding the effects various factors, including fading memory, comparisons are made for all 11 windows for the kinematic and isotropic law and for the first six windows for the Chaboche and B-L law. Standard initial conditions are assumed for all computations. The justification for standard initial conditions is that initial conditions have a different character than the parameters. They depend on the history of material which is unknown. The problem of determining initial conditions by other means is addressed in Section 9. The reason windows seven through eleven are not used for optimization of the Chaboche and B-L laws is that in these windows the use of standard initial conditions has been shown to be inappropriate (see Section 9). After separately computing the vector κ_o of optimal values of parameters for every sample, the optimal vector κ_o is computed for various sets of samples distinguishing between factors of material, history, and window. This optimal vector can be defined in different ways. For example, only the average value could be computed, if it is possible to assume that the distribution function of parameters is Gaussian. The distribution function could be log Gaussian or some other function. The optimal coefficient vector, κ_o is used to predict the stress for a particular law. The reliability coefficient, Θ, for the measured and predicted stress is then computed.

Data in the following tables represent basic statistical data for the reliability coefficient, Θ, error in the difference between predicted and actual stress. A distinction is made between materials and history which means that the average parameters are separately computed for both materials. However, statistical data for the reliability coefficient Θ are combined for both materials. The optimal coefficients are computed for three sets of windows: a) Each separate window, i.e., average the parameters separately in each window so that two sets of optimal coefficients exist for each window one for the 5086 material and one for the 5454 material. b) Average the parameters separately for both materials for windows 1-6 and windows 7-11. The parameters for the appropriate window are used to separately compute the reliability coefficient Θ for windows 1-6 and 7-11. c) For windows 1-6 the reliability coefficient, Θ, is computed with optimal parameters, κ, from the first window and for windows 7-11 the reliability coefficient is computed with the optimal parameters from window 7. Finally the parameters of the

laws and reliability coefficient are computed from available data in handbooks. The procedure for computing these handbook parameters is given in Appendix II.

The quality of the prediction based on the coefficient Θ for other averaging sets was computed. For example, combining history and materials as an averaging set, etc. The results were not essentially different and so they are not reported here. In Tables 7.1-7.3 results are reported for the kinematic law, Tables 7.4-7.6 the isotropic law, Tables 7.7-7.9 the Chaboche law and Table 7.10 the B-L law.

From the reported data the following conclusions can be drawn: a) The kinematic and isotropic laws lead to large discrepancies with experiments for optimal coefficients. These laws are practically unusable for any computational analysis. The error is larger when handbook parameters are used and is of the order of 40% . b) The Chaboche law and the B-L law are essentially of the same quality. Both of these laws have approximately twice the uncertainty of the observed material reproducibility of two samples. (Also see results of Section 9.)

We emphasize that the reported results are only for a one-dimensional case for sheet material. The multidimensional form of the stress strain law, especially for a nonproportional relation between the components of the strain tensor, will contain still larger uncertainty and unreliability [17-19]. This will be especially true when it is not practical to conduct experiments with a large set of samples. Repercussions for the reliability of computational analysis in plasticity are discussed in Section 10.

Table 7.1a. Kinematic law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, separately determined for each of the eleven windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	20.216	20.964	21.668	22.316	23.119	24.281
min	14.088	14.671	15.576	16.044	16.829	17.970
max	25.354	26.325	27.088	28.379	29.119	29.471
std dev	3.166	3.247	3.046	3.018	2.989	3.051

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	20.216	15.272	15.137	14.718	14.288	13.827
min	14.088	11.996	12.054	11.310	10.352	9.960
max	25.354	18.992	18.560	17.818	17.994	17.693
std dev	3.166	1.500	1.510	1.660	1.840	1.867

Table 7.1b. Kinematic law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for windows one through six determined as the average over windows one to six and for windows seven through eleven determined as the average over windows seven through eleven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	21.150	21.76	21.177	22.535	24.174	27.488
min	16.063	15.832	16.062	15.869	18.114	21.084
max	26.647	26.631	27.196	28.734	31.437	34.643
std dev	3.326	3.397	3.185	3.071	3.409	3.492

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	21.150	17.092	15.575	14.778	14.767	17.012
min	16.063	12.079	11.243	11.368	11.052	12.810
max	26.647	19.836	17.740	17.943	18.662	21.041
std dev	3.326	1.739	1.527	1.700	1.887	1.912

Table 7.1c. Kinematic law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for window one through six determined from window one and for windows seven through eleven determined from window seven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	20.216	21.475	23.653	25.892	28.678	32.269
min	14.088	15.403	16.382	18.735	22.164	26.359
max	25.352	27.712	30.670	33.130	35.827	39.024
std dev	3.166	3.219	3.395	3.317	3.284	3.348

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	20.216	15.272	15.716	16.720	18.851	22.058
min	14.088	11.996	12.693	12.814	14.481	18.007
max	25.352	18.992	19.439	21.362	23.588	26.037
std dev	3.166	1.500	1.614	1.865	2.136	1.987

Table 7.2a. Kinematic law. Θ in percent for the random amplitude strain history and κ_o, the optimal parameters, separately determined for each of the eleven windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	22.965	24.079	25.602	26.782	27.836	28.190
min	19.229	18.638	18.996	18.252	19.873	20.954
max	27.992	31.146	31.270	32.466	33.754	34.615
std dev	2.725	3.591	3.491	4.172	3.911	3.773

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	22.965	16.201	13.061	14.160	14.357	15.588
min	19.229	11.957	9.931	10.875	10.795	11.229
max	27.992	22.915	17.617	18.880	17.864	19.763
std dev	2.725	2.600	1.607	1.937	1.419	2.289

Table 7.2b. Kinematic law. Θ in percent for the random amplitude strain history and κ_o, the optimal parameters, for windows one through six determined as the average over windows one to six and for windows seven through eleven determined as the average over windows seven through eleven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	23.431	25.110	26.135	26.791	24.400	28.992
min	17.812	19.262	18.832	19.350	19.451	22.739
max	28.332	31.482	33.247	34.053	34.375	35.226
std dev	2.799	3.258	4.001	4.191	4.036	3.690

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	23.431	18.865	14.102	15.077	14.945	18.203
min	17.812	14.377	10.967	9.986	10.998	12.476
max	28.332	22.785	16.748	19.103	18.444	23.643
std dev	2.799	2.385	1.477	2.468	1.738	2.928

Table 7.2c. Kinematic law. Θ in percent for the random amplitude strain history and κ_o, the optimal parameters, for window one through six determined from window one and for windows seven through eleven determined from window seven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	22.965	24.975	27.388	28.566	29.768	33.178
min	19.229	18.955	20.434	20.149	21.668	24.967
max	27.922	32.775	34.943	36.761	39.774	40.485
std dev	2.725	3.964	4.202	4.248	4.600	4.159

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	22.965	16.201	14.118	16.753	18.508	22.058
min	19.229	11.957	11.507	12.027	13.982	15.866
max	27.922	22.915	17.736	20.222	21.999	28.292
std dev	2.725	2.600	1.931	2.010	1.876	3.172

Table 7.3a. Kinematic law. Θ in percent for the constant amplitude strain history and κ_o, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	35.755	35.303	34.804	34.400	33.887	33.243
min	31.917	31.456	31.066	30.622	30.085	29.459
max	42.255	41.941	41.621	41.350	40.839	39.918
std dev	2.719	2.740	2.748	2.746	2.739	2.658

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	35.755	33.573	31.828	29.762	27.979	26.127
min	31.917	27.181	25.406	23.219	21.967	20.273
max	42.255	39.092	37.292	35.518	34.576	31.933
std dev	2.719	3.118	3.050	3.088	2.996	3.975

Table 7.3b. Kinematic law. Θ in percent for the random amplitude strain history and κ_o, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	35.606	35.219	34.674	34.419	33.889	33.275
min	31.364	30.999	30.485	30.222	29.783	29.047
max	40.133	39.856	39.570	39.354	38.993	38.548
std dev	2.282	2.277	2.281	2.278	2.300	2.329

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	35.606	33.015	30.805	29.730	29.153	28.740
min	31.364	25.667	24.782	24.372	23.209	22.549
max	40.133	39.373	37.192	36.098	35.770	35.785
std dev	2.282	3.448	3.263	2.868	2.928	3.216

Table 7.4a. Isotropic law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, separately determined for each of the eleven windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	20.298	20.889	21.686	22.298	23.210	24.270
min	15.163	16.255	16.731	17.096	17.955	19.154
max	27.953	28.323	28.830	29.433	30.105	30.747
std dev	3.385	3.229	3.161	3.104	3.016	2.891

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	20.298	18.049	17.510	16.775	15.598	15.044
min	15.163	11.918	11.389	11.336	10.502	11.164
max	27.953	22.532	21.780	21.673	20.744	19.378
std dev	3.385	2.270	2.219	2.311	2.346	2.123

Table 7.4b. Isotropic law. Θ in percent for the constant amplitude strain history and κ_O, the optimal parameters, for windows one through six determined as the average over windows one to six and for windows seven through eleven determined as the average over windows seven through eleven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	21.527	21.335	21.727	22.391	23.681	26.236
min	15.872	16.027	17.147	17.522	19.137	20.668
max	29.906	29.658	29.406	29.192	28.792	30.439
std dev	4.140	3.979	3.397	2.948	2.554	2.354

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	21.527	19.924	17.931	16.770	16.631	16.990
min	15.872	13.599	12.174	11.287	12.676	12.197
max	29.906	23.498	22.288	21.657	21.336	21.802
std dev	4.140	2.613	2.261	2.317	2.282	2.471

Table 7.4c. Isotropic law. Θ in percent for the constant amplitude strain history and κ_O, the optimal parameters, for window one through six determined from window one and for windows seven through eleven determined from window seven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	20.298	21.073	22.892	24.630	26.968	30.113
min	15.163	16.576	18.185	18.846	21.537	24.662
max	27.953	27.617	27.376	27.548	31.240	35.295
std dev	3.385	2.837	2.457	2.266	2.500	2.906

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	20.298	18.049	18.179	18.619	19.369	21.641
min	15.163	11.918	13.521	14.678	15.496	16.870
max	27.953	22.532	21.689	22.956	24.576	27.053
std dev	3.385	2.270	2.145	2.284	2.484	2.412

Table 7.5a. Isotropic law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, separately determined for each of the eleven windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	25.943	29.297	30.743	33.833	33.888	36.374
min	20.550	23.433	26.422	25.187	24.836	25.772
max	33.860	35.588	42.816	52.420	51.933	58.692
std dev	3.975	3.494	3.868	6.849	6.931	9.220

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	25.943	25.633	24.442	25.963	24.927	25.076
min	20.550	21.385	19.677	21.349	20.954	19.371
max	33.860	31.388	31.273	28.528	29.881	29.925
std dev	3.975	2.393	2.558	1.910	2.433	2.566

Table 7.5b. Isotropic law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, for windows one through six determined as the average over windows one to six and for windows seven through eleven determined as the average over windows seven through eleven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	27.761	29.567	30.219	32.264	32.447	36.514
min	22.458	23.925	26.085	25.847	25.632	26.789
max	35.993	37.298	40.531	46.278	46.421	54.740
std dev	3.953	3.486	3.666	5.326	5.399	8.663

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	27.761	26.941	24.419	23.753	23.873	25.731
min	22.458	20.440	20.350	15.820	18.688	21.313
max	35.993	32.038	30.185	32.523	29.202	28.669
std dev	3.953	3.174	2.447	3.780	2.095	1.780

Table 7.5c. Isotropic law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, for window one through six determined from window one and for windows seven through eleven determined from window seven.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	25.943	29.048	30.344	31.580	31.943	35.305
min	20.550	22.939	24.961	27.663	27.087	28.556
max	33.860	33.417	33.591	36.990	39.093	46.709
std dev	3.975	3.395	2.064	2.615	3.206	5.652

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	25.943	25.633	28.095	27.443	28.637	31.695
min	20.550	21.385	23.900	24.331	25.531	28.931
max	33.860	31.388	31.864	30.422	33.895	37.721
std dev	3.975	2.393	2.328	1.686	1.845	2.183

Table 7.6a. Isotropic law. Θ in percent for the constant amplitude strain history and κ_O, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	36.700	36.237	35.727	35.315	34.791	34.131
min	32.395	31.893	31.319	30.902	30.413	29.821
max	45.229	44.893	44.552	44.262	43.717	42.735
std dev	3.303	3.314	3.324	3.328	3.321	3.235

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	36.700	33.666	31.949	29.872	28.115	26.187
min	32.395	28.025	26.170	23.851	22.487	20.721
max	45.229	38.947	37.156	35.451	34.451	31.820
std dev	3.303	2.851	2.844	2.925	2.911	2.999

Table 7.6b. Isotropic law. Θ in percent for the random amplitude strain history and κ_o, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	36.456	36.058	35.502	35.241	34.701	34.074
min	31.364	30.999	30.485	30.222	29.783	29.048
max	42.992	42.701	42.395	42.165	41.779	41.305
std dev	2.737	2.739	2.747	2.739	2.758	2.784

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	36.456	33.086	31.004	29.908	29.311	29.156
min	31.364	25.908	24.620	24.417	23.459	22.480
max	42.992	39.404	37.477	36.255	36.001	36.162
std dev	2.737	3.375	3.360	2.951	3.001	3.261

Table 7.7a. Chaboche law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, separately determined for each of the first six windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	11.310	11.471	11.754	12.178	12.347	12.269
min	6.931	7.415	8.336	9.066	9.483	9.346
max	16.547	16.170	15.903	15.799	15.890	15.215
std dev	2.118	1.955	2.072	1.987	1.917	1.689

Table 7.7b. Chaboche law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for windows one through six determined as the average over windows one to six.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	12.246	11.798	11.952	13.157	13.568	13.311
min	8.785	8.520	8.673	10.175	10.987	10.753
max	20.836	20.007	19.027	18.485	18.134	17.692
std dev	2.434	2.348	2.038	1.932	1.699	1.678

Table 7.7c. Chaboche law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for window one through six determined from window one.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	11.310	12.250	16.922	18.863	19.045	18.655
min	6.931	8.391	13.383	14.793	15.072	14.726
max	16.547	15.847	19.825	22.965	23.347	21.886
std dev	2.118	1.761	2.020	2.198	2.159	2.134

Table 7.8a. Chaboche law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, separately determined for each of the first six windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	13.297	13.216	13.234	13.776	13.057	14.478
min	8.742	8.469	10.325	10.306	9.140	10.754
max	18.925	17.911	17.801	17.372	17.358	18.235
std dev	2.569	2.324	2.046	1.832	2.194	1.819

Table 7.8b. Chaboche law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, for windows one through six determined as the average over windows one to six.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	13.592	13.698	13.521	13.293	12.961	15.097
min	10.024	9.921	9.665	9.458	9.034	11.412
max	17.611	17.550	18.151	17.822	17.301	19.026
std dev	2.069	2.030	2.055	2.012	2.132	1.725

Table 7.8c. Chaboche law. Θ in percent for the random amplitude strain history and κ_O, the optimal parameters, for window one through six determined from window one.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	13.297	13.672	14.038	14.104	13.687	13.568
min	8.742	9.018	9.026	9.515	9.097	10.095
max	18.925	18.784	20.107	20.111	19.407	18.473
std dev	2.569	2.222	2.257	2.304	2.144	1.959

Table 7.9a. Chaboche law. Θ in percent for the constant amplitude strain history and κ_O, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	25.609	25.119	24.245	23.371	22.394	21.367
min	21.005	20.497	19.804	19.048	18.214	17.345
max	34.625	33.870	32.759	31.700	30.528	29.012
std dev	3.170	3.133	3.034	2.948	2.851	2.692

Table 7.9b. Chaboche law. Θ in percent for the random amplitude strain history and κ_O, the parameters, determined from published data in the literature and in handbooks.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	26.033	25.675	25.181	24.850	24.244	23.251
min	20.570	20.304	19.928	19.607	19.052	18.048
max	31.971	31.672	31.237	30.812	30.091	29.187
std dev	3.092	3.073	3.118	3.068	2.926	2.867

Table 7.10a. B-L law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, separately determined for each of the first six windows.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	9.753	9.956	10.438	11.111	11.374	11.239
min	5.354	6.431	7.118	7.873	9.138	8.679
max	15.111	15.234	14.995	15.360	15.230	15.098
std dev	2.257	2.205	1.969	1.768	1.544	1.532

Table 7.10b. B-L law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for windows one through six determined as the average over windows one to six.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	11.032	10.611	10.840	12.387	12.844	12.600
min	6.650	6.360	8.203	9.649	10.341	10.123
max	16.553	15.903	15.139	15.275	14.984	15.040
std dev	2.548	2.438	1.764	1.646	1.473	1.465

Table 7.10c. B-L law. Θ in percent for the constant amplitude strain history and κ_o, the optimal parameters, for window one through six determined from window one.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	9.735	10.458	14.590	15.896	15.709	15.406
min	5.354	6.753	11.389	12.773	12.592	12.319
max	15.111	14.471	17.865	18.679	18.287	17.954
std dev	2.257	1.831	1.905	1.769	1.688	1.665

8. THE CONSTITUTIVE LAW PARAMETERS AND THEIR ANALYSIS

As was explained in Section 7 the parameters of the constitutive law are determined by an optimal fitting procedure for standard initial conditions. The optimal initial conditions are commented on in Section 9. For the kinematic and isotropic law, parameters are determined for all 11 windows and cyclic and random strain histories. For the Chaboche law parameters are computed for windows 1-6 and cyclic and random histories. Finally, for the B-L law parameters are computed for windows 1-6 and cyclic strain histories.

In this section the parameters will be analyzed statistically. Note that in Section 5 only the influence of various factors such as history, mean level, material and window on the reproducibility of material response was analyzed. No influence of a factor means that the reproducibility is not influenced, but this does not mean that in general the material behavior is not influenced by the factor. In this section we will analyze the difference between actual material behavior and the predicted behavior from constitutive laws which depend on the parameters.

1) The Kinematic Law. In Tables 8.1a-c and 8.2a-c we report values of the parameters (E, E_p, σ_y) for cyclic and random strain histories. The mean and standard deviation for the 5086 material (ave 1, std dev 1) and the 5454 material (ave 2, std dev 2) are reported along with the mean and standard deviation when no distinction is made between materials (ave, std dev).

An analysis of variance of these results is reported in Tables 8.3a-c for a $p = 0.05$ level for all cases. The notation in rows 1-7 of Table 8.3a is the following: Row 1) If * is reported then the value of E depends significantly on history. This means that the null hypothesis that E is not influenced by history can be rejected at a $p = 0.05$ level. More precisely, if the null hypothesis were true there would be at most a 5% chance of observing differences of E values for cyclic and random strain histories as large as the differences actually computed from the data. Since these observed differences would be unusual under the null hypothesis, we conclude that the null hypothesis must be false. If – is reported we cannot claim that the null hypothesis is false, hence it could be correct and no influence of history is observed. The above statements refer to the mean of the parameters over all levels and materials. Row 2) The influence of material reported in the row 2 has analogous meaning. Here we refer to averages of the coefficients over all mean strain levels and both types of strain histories. Row 3) If * is reported then

dependence of E on the history factor (cyclic or random strain) is influenced by the material. In other words, if we separately examine the dependence of E on history for material 1 (5086) and material 2 (5454) we would observe that the dependence is different for each material. The null hypothesis that there is no influence of material on the dependence of E on history can be rejected at a $p = 0.05$ level. If – is reported the null hypotheses cannot be rejected and it could be correct that no influence of material on the history factor is observed. Equivalently, if * is reported in the table we can say that dependence of E on the material factor is influenced by the history, i.e., the conclusions are commutative. These statements refer to averages at all mean strain levels. Row 4) The interpretation in terms of mean strain level is analogous to that of row 1). Rows 5-6). The interpretation for history/level and material/level influences is analogous to that of row 3). Row 7) If * is reported then dependence of E on history and material is influenced by the mean strain level. If we separately examine the dependence of E on history and material for each mean strain level, we would observe that the dependence is different for different mean strain levels. Equivalently, the table indicates the relative influence of one factor on the two others, i.e., the conclusions are commutative.

In Table 8.4 we report the dependence of parameters E, E_p, σ_y on the window. The interpretation of * and – is the same as in Table 8.3 with the exception that there is no communtativity with respect to w because measurements taken on different windows are related since these measurements are made on the same material sample. For example, the first row indicates an influence of w on E and σ_y when all data for different histories, materials and mean levels are averaged. This influence is due to the rapid strain hardening which occurs at the beginning of the strain history. Note that in windows 7-11 the standard initial conditions (i.e., $\alpha(0) = 0$) are used. The effect of this assumption will be addressed in Section 9. It could be expected that there is an influence of material. But it is interesting that there is quite a significant influence of history on all parameters, which suggests that there is an effect of history on material behavior. In fact, this is true for all four constitutive models studied. The same conclusions were obtained by various other statistical tests such as Wilk's Lambda, Pillai's Trace and Hotelling-Lawley Trace.

From the results reported we can draw the following interesting conclusions: a) The value of modulus of elasticity depends on history for all windows. The differences in modulus of elasticity are of the order of 3%. The effect of material is also clearly significant. It is interesting that the influence of mean strain level disappears in windows 7-11, which indicates that this effect occurs in the initial cycles of the strain history.
b) The modulus of plasticity depends on history especially in windows 7-11 with

variations on the order of 10%. c) The yield strain depends on history with variations on the order of 1-3%. We emphasize that these conclusions do not necessarily coincide with the true modulus of elasticity, plasticity and yield strain, since these are not uniquely defined either. These conclusions should not be taken out of context with the model under consideration and method of parameter identification.

In Tables 8.1 and 8.2 are reported the mean and standard deviation of the parameters. However, this does not suggest that E, E_p, and σ_y have a normal probability density. The correlation matrix is shown in Table 8.5 for window 6 (0-500) when no distinction is made between histories. This correlation indicates that the parameters are not totally independent of each other. The correlation matrix can be used for Monte Carlo simulations to investigate the sensitivity of a constitutive law to variations in parameters of the law. Because the data are positive the covariance and correlation matrix could also be used for log E, log E_p and log S_y. The level of significance of the correlation coefficients is not reported.

2) The Isotropic Law. Tables 8.6 - 8.10 are analogous to those of the kinematic law and similar features are observed. For example, the dependence of the parameters on history is similar. The modulus of elasticity varies by about 10%, between the cyclic and random strain history. The parameter E_p, has a large coefficient of variation and the yield stress varies by 10%. The dependence of E_p on history is quite different for the kinematic and isotropic law. As we have seen in Section 7 both the kinematic and isotropic laws, are very unreliable.

3) The Chaboche Law. Tables 8.11 - 8.15 are analogous to those described previously for the kinematic and isotropic laws. Only windows 1-6, when the material is in the "virgin" or as received state, are reported because of computational considerations. As seen in Section 7 the Chaboche law is much more realistic. This fact is also visible in Table 8.13a where the modulus of elasticity is essentially only dependent on material and not on the other factors. This is to be expected since the modulus of elasticity is the most stable material property considered. The other parameters are more or less strongly dependent on the other factors considered. In Table 8.14 we see a dependence of a, b and ε_y on the window. Once more all factors of history, material and level considered together play a significant role in influencing the parameters.

4) The B-L Law. The analysis for the B-L law is reported in an analogous format. The influence of history (windows 7-11) is not reported because the computation of

optimal coefficients is expensive due to the presence of many local minima. In Tables 8.16-8.18 the data are reported as was done in the previous models. As for the Chaboche law the modulus of elasticity is the most stable parameters.

In Table 8.20 we compare the values of modulus of elasticity computed from the four constitutive laws. The modulus of elasticity in the laws is constant and does not take into consideration any changes due to the influence of cyclic plasticity on the structure of the material. Also the modulus values reported in Table 8.20 do not agree in general with modulus values determined solely in the elastic region. This results from the fact that the parameters are determined as a best fit by the error measure Θ over a range of elastic and plastic strains.

Table 8.1a. Kinematic law modulus of elasticity, E, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.8868D+07	0.8933D+07	0.9007D+07	0.9081D+07	0.9161D+07	0.9252D+07
ave 1	0.8828D+07	0.8890D+07	0.8973D+07	0.9040D+07	0.9116D+07	0.9201D+07
ave 2	0.8907D+07	0.8975D+07	0.9041D+07	0.9122D+07	0.9206D+07	0.9302D+07
std dev	0.5256D+06	0.5198D+06	0.4900D+06	0.4881D+06	0.4725D+06	0.4593D+06
std dev 1	0.5433D+06	0.5318D+06	0.5201D+06	0.5173D+06	0.5162D+06	0.5248D+06
std dev 2	0.5041D+06	0.5040D+06	0.4554D+06	0.4534D+06	0.4196D+06	0.3760D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.8868D+07	0.9054D+07	0.9092D+07	0.9152D+07	0.9188D+07	0.9235D+07
ave 1	0.8828D+07	0.9033D+07	0.9075D+07	0.9137D+07	0.9186D+07	0.9233D+07
ave 2	0.8907D+07	0.9075D+07	0.9109D+07	0.9167D+07	0.9190D+07	0.9236D+07
std dev	0.5256D+06	0.2955D+06	0.2896D+06	0.2832D+06	0.2804D+06	0.2632D+06
std dev 1	0.5433D+06	0.3554D+06	0.3342D+06	0.3164D+06	0.3075D+06	0.2784D+06
std dev 2	0.5041D+06	0.2177D+06	0.2355D+06	0.2447D+06	0.2502D+06	0.2471D+06

Table 8.1b. Kinematic law modulus of plasticity, E_p, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.7231D+06	0.7353D+06	0.7351D+06	0.7546D+06	0.7812D+06	0.8371D+06
ave 1	0.7139D+06	0.7167D+06	0.7234D+06	0.7230D+06	0.7593D+06	0.8762D+06
ave 2	0.7322D+06	0.7539D+06	0.7469D+06	0.7862D+06	0.8032D+06	0.7980D+06
std dev	0.8998D+06	0.9306D+06	0.9038D+06	0.9083D+06	0.9198D+06	0.1026D+07
std dev 1	0.9263D+06	0.9415D+06	0.9174D+06	0.8877D+06	0.9023D+06	0.1112D+07
std dev 2	0.8724D+06	0.9193D+06	0.3899D+06	0.9274D+06	0.9365D+06	0.9300D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.7231D+06	0.1906D+07	0.1930D+07	0.2054D+07	0.1997D+07	0.1954D+07
ave 1	0.7139D+06	0.2073D+07	0.2104D+07	0.2302D+07	0.2272D+07	0.2234D+07
ave 2	0.7322D+06	0.1740D+07	0.1755D+07	0.1807D+07	0.1722D+07	0.1674D+07
std dev	0.8998D+06	0.4231D+06	0.4422D+06	0.4697D+06	0.5026D+06	0.5504D+06
std dev 1	0.9263D+06	0.4866D+06	0.5060D+06	0.4357D+06	0.4695D+06	0.5325D+06
std dev 2	0.8724D+06	0.2527D+06	0.2726D+06	0.3590D+06	0.3650D+06	0.4069D+06

Table 8.1c. Kinematic law yield stress, σ_y, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2876D+05	0.2936D+05	0.3011D+05	0.3069D+05	0.3137D+05	0.3223D+05
ave 1	0.2954D+05	0.3022D+05	0.3103D+05	0.3169D+05	0.3235D+05	0.3307D+05
ave 2	0.2799D+05	0.2849D+05	0.2919D+05	0.2969D+05	0.3040D+05	0.3138D+05
std dev	0.1886D+04	0.2071D+04	0.2178D+04	0.2265D+04	0.2353D+04	0.2664D+04
std dev 1	0.1744D+04	0.1889D+04	0.2016D+04	0.1992D+04	0.2133D+04	0.2859D+04
std dev 2	0.1697D+04	0.1877D+04	0.1930D+04	0.2068D+04	0.2148D+04	0.2146D+04

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.2876D+05	0.2969D+05	0.3076D+05	0.3163D+05	0.3289D+05	0.3455D+05
ave 1	0.2954D+05	0.3074D+05	0.3198D+05	0.3285D+05	0.3411D+05	0.3586D+05
ave 2	0.2799D+05	0.2865D+05	0.2954D+05	0.3041D+05	0.3167D+05	0.3324D+05
std dev	0.1886D+04	0.1182D+04	0.1356D+04	0.1371D+04	0.1387D+04	0.1486D+04
std dev 1	0.1744D+04	0.5610D+03	0.5835D+03	0.5900D+03	0.6905D+03	0.8579D+03
std dev 2	0.1697D+04	0.5286D+03	0.5993D+03	0.6569D+03	0.6260D+03	0.5017D+03

Table 8.2a. Kinematic law modulus of elasticity, *E*, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9228D+07	0.9356D+07	0.9453D+07	0.9556D+07	0.9687D+07	0.9753D+07
ave 1	0.9017D+07	0.9193D+07	0.9307D+07	0.9295D+07	0.9468D+07	0.9485D+07
ave 2	0.9438D+07	0.9520D+07	0.9598D+07	0.9817D+07	0.9906D+07	0.1002D+08
std dev	0.3850D+06	0.3177D+06	0.2961D+06	0.4353D+06	0.3863D+06	0.3803D+06
std dev 1	0.3364D+06	0.2705D+06	0.2502D+06	0.2731D+06	0.2564D+06	0.2286D+06
std dev 2	0.3075D+06	0.2740D+06	0.2655D+06	0.4094D+06	0.3696D+06	0.3039D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.9228D+07	0.9499D+07	0.9699D+07	0.9549D+07	0.9531D+07	0.9746D+07
ave 1	0.9017D+07	0.9361D+07	0.9388D+07	0.9359D+07	0.9389D+07	0.9603D+07
ave 2	0.9438D+07	0.9637D+07	0.1001D+08	0.9739D+07	0.9674D+07	0.9889D+07
std dev	0.3850D+06	0.2753D+06	0.4591D+06	0.3465D+06	0.3990D+06	0.3112D+06
std dev 1	0.3364D+06	0.1877D+06	0.3528D+06	0.2469D+06	0.4095D+06	0.2565D+06
std dev 2	0.3075D+06	0.2797D+06	0.3220D+06	0.3268D+06	0.3319D+06	0.2948D+06

Table 8.2b. Kinematic law modulus of plasticity, E_p, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9196D+06	0.8235D+06	0.7796D+06	0.8117D+06	0.7766D+06	0.7712D+06
ave 1	0.8728D+06	0.8135D+06	0.8090D+06	0.8580D+06	0.7940D+06	0.7910D+06
ave 2	0.9664D+06	0.8336D+06	0.7501D+06	0.7653D+06	0.7591D+06	0.7513D+06
std dev	0.5943D+06	0.5789D+06	0.7070D+06	0.8480D+06	0.7663D+06	0.7978D+06
std dev 1	0.5786D+06	0.6162D+06	0.7768D+06	0.9109D+06	0.8118D+06	0.8657D+06
std dev 2	0.6061D+06	0.5388D+06	0.6281D+06	0.7773D+06	0.7175D+06	0.7231D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.9196D+06	0.2776D+07	0.3488D+07	0.2719D+07	0.3371D+07	0.3020D+07
ave 1	0.8728D+06	0.3037D+07	0.3690D+07	0.2997D+07	0.3553D+07	0.3192D+07
ave 2	0.9664D+06	0.2516D+07	0.3287D+06	0.2441D+07	0.3189D+07	0.2849D+07
std dev	0.5943D+06	0.3167D+06	0.4134D+06	0.3642D+06	0.4280D+06	0.2638D+06
std dev 1	0.5786D+06	0.1817D+06	0.3821D+06	0.2701D+06	0.4816D+06	0.2559D+06
std dev 2	0.6061D+06	0.1786D+06	0.3381D+06	0.1948D+06	0.2610D+06	0.1220D+06

Table 8.2c. Kinematic law yield stress, σ_y, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2741D+05	0.2755D+05	0.2879D+05	0.3019D+05	0.3128D+05	0.3225D+05
ave 1	0.2820D+05	0.2861D+05	0.3084D+05	0.3131D+05	0.3279D+05	0.3358D+05
ave 2	0.2661D+05	0.2649D+05	0.2674D+05	0.2907D+05	0.2976D+05	0.3091D+05
std dev	0.1830D+04	0.1984D+04	0.2592D+04	0.2264D+04	0.2588D+04	0.2536D+04
std dev 1	0.2000D+04	0.2276D+04	0.2147D+04	0.2000D+04	0.2641D+04	0.2569D+04
std dev 2	0.1195D+04	0.6778D+03	0.6456D+03	0.1933D+04	0.1346D+04	0.1646D+04

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.2741D+05	0.2622D+05	0.2691D+05	0.2832D+05	0.2782D+05	0.3075D+05
ave 1	0.2820D+05	0.2667D+05	0.2797D+05	0.2845D+05	0.2847D+05	0.3278D+05
ave 2	0.2661D+05	0.2577D+05	0.2585D+05	0.2820D+05	0.2718D+05	0.2872D+05
std dev	0.1830D+04	0.7203D+03	0.1933D+04	0.7403D+03	0.1360D+04	0.2203D+04
std dev 1	0.2000D+04	0.5065D+03	0.1952D+04	0.6932D+03	0.1397D+04	0.9823D+03
std dev 2	0.1195D+04	0.6071D+03	0.1186D+04	0.7652D+03	0.9571D+03	0.7130D+03

Table 8.3a. Kinematic law significance of the influence of various factors on the modulus of elasticity, E.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	*	-	-	*	*	*
L	*	*	*	*	*	*
HxL	*	*	*	*	*	*
MxL	*	*	-	*	*	*
HxMxL	-	*	*	*	*	*

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	*	*	*	*	*	*
L	*	-	-	-	-	-
HxL	*	*	*	*	*	-
MxL	*	*	*	*	*	-
HxMxL	-	*	*	*	*	*

Table 8.3b. Kinematic law significance of the influence of various factors on the modulus of plasticity, E_p.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	-	-	-	-
M	-	-	-	-	-	-
HxM	-	-	-	-	-	-
L	*	*	*	*	*	*
HxL	*	*	*	*	*	*
MxL	*	*	*	*	*	*
HxMxL	-	*	*	-	-	*

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	*	*	*	*
M	-	*	*	*	*	*
HxM	-	*	-	-	-	-
L	*	*	*	*	*	*
HxL	*	*	*	*	*	*
MxL	*	*	*	-	-	-
HxMxL	-	-	-	-	-	-

Table 8.3c. Kinematic law significance of the influence of various factors on the yield stress, σ_y.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	-	-
M	*	*	*	*	*	*
HxM	-	-	*	-	-	-
L	*	*	*	*	*	*
HxL	*	*	*	*	*	-
MxL	*	*	*	*	-	-
HxMxL	*	*	*	*	-	-

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	-	*	-	*	*	*
L	*	-	*	-	-	-
HxL	*	-	*	-	-	-
MxL	*	-	*	*	-	-
HxMxL	*	*	*	-	-	-

Table 8.4. Kinematic law significance of the dependence of the parameters on the window.

Parameter	E	E_p	σ_y
W	*	-	*
WxH	*	*	*
WxM	*	-	*
WxHxM	*	-	*
WxL	*	*	*
WxLxH	*	*	*
WxLxM	*	*	*
WxLxHxM	*	*	*

Table 8.5. Kinematic law correlation matrix for window six.

	5086			5454			5086 and 5454		
	E	E_p	σ_y	E	E_p	σ_y	E	E_p	σ_y
E	1.000	0.459	-0.123	1.000	0.372	-0.304	1.000	0.377	-0.306
E_p	0.459	1.000	-0.825	0.372	1.000	-0.731	0.377	1.000	-0.701
σ_y	-0.123	-0.825	1.000	-0.304	-0.731	1.000	-0.306	-0.701	1.000

Table 8.6a. Isotropic law modulus of elasticity, E, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.8593D+07	0.8658D+07	0.8743D+07	0.8812D+07	0.8906D+07	0.9022D+07
ave 1	0.8530D+07	0.8615D+07	0.8697D+07	0.8749D+07	0.8832D+07	0.8936D+07
ave 2	0.8656D+07	0.8702D+07	0.8790D+07	0.8874D+07	0.8979D+07	0.9108D+07
std dev	0.3371D+06	0.3314D+06	0.3397D+06	0.3522D+06	0.3699D+06	0.3768D+06
std dev 1	0.3108D+06	0.3325D+06	0.3448D+06	0.3506D+06	0.3742D+06	0.3900D+06
std dev 2	0.3503D+06	0.3245D+06	0.3281D+06	0.3425D+06	0.3503D+06	0.3423D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.8593D+07	0.8815D+07	0.8878D+07	0.8962D+07	0.9033D+07	0.9131D+07
ave 1	0.8530D+07	0.8725D+07	0.8798D+07	0.8901D+07	0.8991D+07	0.9112D+07
ave 2	0.8656D+07	0.8905D+07	0.8958D+07	0.9023D+07	0.9074D+07	0.9151D+07
std dev	0.3371D+06	0.2699D+06	0.2685D+06	0.2704D+06	0.2732D+06	0.2668D+06
std dev 1	0.3108D+06	0.2655D+06	0.2597D+06	0.2643D+06	0.2747D+06	0.2787D+06
std dev 2	0.3503D+06	0.2430D+06	0.2529D+06	0.2626D+06	0.2652D+06	0.2529D+06

Table 8.6b. Isotropic law modulus of plasticity, E_p, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1264D+06	0.1343D+06	0.1515D+06	0.1696D+06	0.2180D+06	0.2738D+06
ave 1	0.4551D+05	0.5280D+05	0.5184D+05	0.6088D+05	0.1153D+06	0.1905D+06
ave 2	0.2072D+06	0.2157D+06	0.2512D+06	0.2784D+06	0.3208D+06	0.3570D+06
std dev	0.1928D+06	0.1740D+06	0.1768D+06	0.1808D+06	0.1773D+06	0.1786D+06
std dev 1	0.7740D+05	0.8171D+05	0.9163D+05	0.7926D+05	0.9973D+05	0.1549D+06
std dev 2	0.2352D+06	0.2015D+06	0.1850D+06	0.1883D+06	0.1783D+06	0.1611D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.1264D+06	0.7020D+06	0.7056D+06	0.6972D+06	0.6863D+06	0.6739D+06
ave 1	0.4551D+05	0.5164D+06	0.5019D+06	0.4936D+06	0.4783D+06	0.4876D+06
ave 2	0.2072D+06	0.8875D+06	0.9093D+06	0.9008D+06	0.8944D+06	0.8602D+06
std dev	0.1928D+06	0.2503D+06	0.2622D+06	0.2605D+06	0.2763D+06	0.2829D+06
std dev 1	0.7740D+05	0.1644D+06	0.1580D+06	0.1666D+06	0.2055D+06	0.2417D+06
std dev 2	0.2352D+06	0.1715D+06	0.1719D+06	0.1584D+06	0.1545D+06	0.1794D+06

Table 8.6c. Isotropic law yield stress, σ_y, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2887D+05	0.2938D+05	0.2999D+05	0.3043D+05	0.3086D+05	0.3139D+05
ave 1	0.2976D+05	0.3047D+05	0.3126D+05	0.3175D+05	0.3214D+05	0.3252D+05
ave 2	0.2798D+05	0.2830D+05	0.2871D+05	0.2912D+05	0.2959D+05	0.3026D+05
std dev	0.1216D+04	0.1420D+04	0.1687D+04	0.1797D+04	0.1938D+04	0.2187D+04
std dev 1	0.7329D+03	0.7516D+03	0.8152D+03	0.8537D+03	0.1144D+04	0.1734D+04
std dev 2	0.9185D+03	0.1058D+04	0.1335D+04	0.1510D+04	0.1714D+04	0.2000D+04

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.2887D+05	0.2947D+05	0.3072D+05	0.3197D+05	0.3312D+05	0.3468D+05
ave 1	0.2976D+05	0.3155D+05	0.3294D+05	0.3438D+05	0.3561D+05	0.3701D+05
ave 2	0.2798D+05	0.2740D+05	0.2850D+05	0.2956D+05	0.3063D+05	0.3234D+05
std dev	0.1216D+04	0.2401D+04	0.2434D+04	0.2582D+04	0.2644D+04	0.2493D+04
std dev 1	0.7329D+03	0.8606D+03	0.6706D+03	0.7497D+03	0.8079D+03	0.7679D+03
std dev 2	0.9185D+03	0.1472D+04	0.1251D+04	0.1086D+04	0.9423D+03	0.9804D+03

Table 8.7a. Isotropic law modulus of elasticity, E, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9022D+07	0.9252D+07	0.9523D+07	0.9688D+07	0.9718D+07	0.9817D+07
ave 1	0.8805D+07	0.9098D+07	0.9455D+07	0.9444D+07	0.9460D+07	0.9575D+07
ave 2	0.9238D+07	0.9406D+07	0.9590D+07	0.9933D+07	0.9976D+07	0.1006D+08
std dev	0.4676D+06	0.4026D+06	0.3032D+06	0.3830D+06	0.4275D+06	0.4128D+06
std dev 1	0.3778D+06	0.3264D+06	0.2595D+06	0.2321D+06	0.2948D+06	0.3190D+06
std dev 2	0.4482D+06	0.4124D+06	0.3278D+06	0.3460D+06	0.3814D+06	0.3495D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.9022D+07	0.9272D+07	0.9562D+07	0.9467D+07	0.9509D+07	0.9827D+07
ave 1	0.8805D+07	0.9157D+07	0.9258D+07	0.9243D+07	0.9299D+07	0.9621D+07
ave 2	0.9238D+07	0.9387D+07	0.9867D+07	0.9691D+07	0.9720D+07	0.1003D+08
std dev	0.4676D+06	0.4143D+06	0.4577D+06	0.4475D+06	0.4170D+06	0.3898D+06
std dev 1	0.3778D+06	0.3906D+06	0.3303D+06	0.3713D+06	0.3513D+06	0.3370D+06
std dev 2	0.4482D+06	0.4053D+06	0.3519D+06	0.4031D+06	0.3687D+06	0.3249D+06

Table 8.7b. Isotropic law modulus of plasticity, E_p, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.5461D+06	0.7303D+06	0.1036D+07	0.1314D+07	0.1343D+07	0.1299D+07
ave 1	0.5140D+06	0.7586D+06	0.1106D+07	0.1228D+07	0.1258D+07	0.1189D+07
ave 2	0.5782D+06	0.7021D+06	0.9659D+06	0.1401D+07	0.1427D+07	0.1408D+07
std dev	0.3105D+06	0.3260D+06	0.4867D+06	0.5395D+06	0.6206D+06	0.6274D+06
std dev 1	0.1775D+06	0.2915D+06	0.5987D+06	0.6189D+06	0.7224D+06	0.7486D+06
std dev 2	0.3991D+06	0.3550D+06	0.3248D+06	0.4291D+06	0.4838D+06	0.4428D+06

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.5461D+06	0.1083D+07	0.9670D+06	0.1163D+07	0.1329D+07	0.1869D+07
ave 1	0.5140D+06	0.1527D+07	0.9266D+06	0.1112D+07	0.1513D+07	0.2033D+07
ave 2	0.5782D+06	0.6392D+06	0.1008D+07	0.1213D+07	0.1145D+07	0.1705D+07
std dev	0.3105D+06	0.7677D+06	0.9158D+06	0.5045D+06	0.5703D+06	0.6324D+06
std dev 1	0.1775D+06	0.7223D+06	0.1044D+07	0.5685D+06	0.5816D+06	0.5788D+06
std dev 2	0.3991D+06	0.5125D+06	0.7641D+06	0.4250D+06	0.4944D+06	0.6411D+06

Table 8.7c. Isotropic law yield stress, σ_y, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2680D+05	0.2649D+05	0.2704D+05	0.2707D+05	0.2721D+05	0.2825D+05
ave 1	0.2679D+05	0.2672D+05	0.2912D+05	0.2874D+05	0.2843D+05	0.3001D+05
ave 2	0.2681D+05	0.2625D+05	0.2495D+05	0.2539D+05	0.2599D+05	0.2650D+05
std dev	0.2623D+04	0.3750D+04	0.5083D+04	0.5685D+04	0.6258D+04	0.6415D+04
std dev 1	0.3014D+04	0.3922D+04	0.4935D+04	0.4848D+04	0.5916D+04	0.6089D+04
std dev 2	0.2164D+04	0.3554D+04	0.4315D+04	0.5962D+04	0.6352D+04	0.6248D+04

Window	0-10	10-20	20-50	50-100	100-200	200-500
ave	0.2680D+05	0.2565D+05	0.3186D+05	0.3348D+05	0.3337D+05	0.3068D+05
ave 1	0.2679D+05	0.2516D+05	0.3320D+05	0.3591D+05	0.3895D+05	0.3048D+05
ave 2	0.2681D+05	0.2614D+05	0.3051D+05	0.3106D+05	0.2779D+05	0.3088D+05
std dev	0.2623D+04	0.2088D+04	0.4836D+04	0.9221D+04	0.1609D+05	0.2645D+04
std dev 1	0.3014D+04	0.2414D+04	0.6106D+04	0.1197D+05	0.2132D+05	0.2183D+04
std dev 2	0.2164D+04	0.1553D+04	0.2422D+04	0.3862D+04	0.1046D+04	0.3024D+04

Table 8.8a. Isotropic law significance of the influence of various factors on the modulus of elasticity, E.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	*
M	*	*	-	*	*	*
HxM	*	-	-	*	*	*
L	*	*	*	*	*	*
HxL	*	*	-	-	-	-
MxL	*	*	*	*	*	*
HxMxL	*	*	-	-	-	*

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	*	-	*	*	*	*
L	*	*	*	*	*	*
HxL	*	*	*	-	*	*
MxL	*	*	*	-	*	*
HxMxL	*	*	*	-	*	*

Table 8.8b. Isotropic law significance of the influence of various factors on the modulus of plasticity, E_p.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	*
M	*	-	-	-	-	-
HxM	-	-	-	-	-	-
L	*	*	*	*	*	*
HxL	*	*	-	*	*	*
MxL	*	-	-	-	-	-
HxMxL	*	-	-	-	-	-

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	*	*	*	*
M	*	*	*	*	-	-
HxM	-	*	-	-	*	*
L	*	*	*	-	-	-
HxL	*	*	*	-	*	*
MxL	*	-	-	-	-	-
HxMxL	*	-	-	-	-	-

Table 8.8c. Isotropic law significance of the influence of various factors on the yield stress, σ_y.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	*	*	-	-	-	-
L	*	*	*	*	*	*
HxL	*	*	*	*	-	-
MxL	*	-	-	-	-	-
HxMxL	*	*	-	-	-	-

Window	0-10	10-20	20-50	50-100	100-200	200-500
H	*	*	-	-	-	*
M	*	*	*	*	*	*
HxM	*	*	-	-	-	*
L	*	-	-	-	-	-
HxL	*	*	-	-	-	-
MxL	*	-	-	-	-	-
HxMxL	*	-	-	-	-	-

Table 8.9. Isotropic law significance of the dependence of the parameters on the window.

Parameter	E	E_p	σ_y
W	*	*	*
WxH	*	*	-
WxM	*	-	*
WxHxM	*	-	*
WxL	*	*	*
WxLxH	-	*	-
WxLxM	-	-	-
WxLxHxM	-	-	-

Table 8.10. Isotropic law correlation matrix for window six.

	5086			5454			5086 and 5454		
	E	E_p	σ_y	E	E_p	σ_y	E	E_p	σ_y
E	1.000	0.586	-0.195	1.000	0.649	0.021	1.000	0.616	-0.149
E_p	0.586	1.000	-0.719	0.576	1.000	-0.574	0.616	1.000	-0.652
σ_y	-0.719	-0.195	1.000	0.021	-0.574	1.000	-0.149	-0.652	1.000

Table 8.11a. Chaboche law modulus of elasticity, *E*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1001D+08	0.1001D+08	0.9976D+07	0.9931D+07	0.9934D+07	0.9890D+07
ave 1	0.9857D+07	0.9904D+07	0.9806D+07	0.9763D+07	0.9762D+07	0.9813D+07
ave 2	0.1016D+08	0.1011D+08	0.1015D+08	0.1010D+08	0.1011D+08	0.9967D+07
std dev	0.3719D+06	0.4049D+06	0.3886D+06	0.3317D+06	0.3336D+06	0.3308D+06
std dev 1	0.3893D+06	0.4704D+06	0.4247D+06	0.3648D+06	0.3758D+06	0.4131D+06
std dev 2	0.2839D+06	0.2905D+06	0.2517D+06	0.1749D+06	0.1496D+06	0.1906D+06

Table 8.11b. Chaboche law yield strain, ε_y, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2165D-02	0.2136D-02	0.2118D-02	0.2122D-02	0.2080D-02	0.2076D-02
ave 1	0.2157D-02	0.2086D-02	0.2119D-02	0.2129D-02	0.2075D-02	0.2026D-02
ave 2	0.2173D-02	0.2186D-02	0.2117D-02	0.2116D-02	0.2085D-02	0.2125D-02
std dev	0.1840D-03	0.2104D-03	0.2159D-03	0.1954D-03	0.2392D-03	0.2648D-03
std dev 1	0.2083D-03	0.2499D-03	0.2498D-03	0.2116D-03	0.2765D-03	0.2827D-03
std dev 2	0.1555D-03	0.1450D-03	0.1755D-03	0.1776D-03	0.1946D-03	0.2355D-03

Table 8.11c. Chaboche law kinematic coefficient, *a*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9842D+04	0.1067D+05	0.1128D+05	0.1154D+05	0.1222D+05	0.1238D+05
ave 1	0.1037D+05	0.1142D+05	0.1233D+05	0.1250D+05	0.1325D+05	0.1386D+05
ave 2	0.9313D+04	0.9920D+04	0.1023D+05	0.1057D+05	0.1120D+05	0.1089D+05
std dev	0.1738D+04	0.1960D+04	0.2229D+04	0.1965D+04	0.2134D+04	0.2386D+04
std dev 1	0.1912D+04	0.2292D+04	0.2076D+04	0.1695D+04	0.2133D+04	0.2000D+04
std dev 2	0.1352D+04	0.1138D+04	0.1853D+04	0.1725D+04	0.1566D+04	0.1721D+04

Table 8.11d. Chaboche law isotropic exponent, *b*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.5111D+01	0.3679D+01	0.2224D+01	0.1608D+01	0.1202D+01	0.9292D+00
ave 1	0.5694D+01	0.4083D+01	0.2487D+01	0.1840D+01	0.1386D+01	0.1208D+01
ave 2	0.4529D+01	0.3275D+01	0.1960D+01	0.1376D+01	0.1018D+01	0.6500D+00
std dev	0.1146D+01	0.8656D+00	0.7092D+00	0.6552D+00	0.6812D+00	0.6247D+00
std dev 1	0.9530D+00	0.6387D+00	0.4870D+00	0.4292D+00	0.5633D+00	0.6326D+00
std dev 2	0.1019D+01	0.8738D+00	0.7935D+00	0.7529D+00	0.7368D+00	0.4737D+00

Table 8.11e. Chaboche law kinematic exponent, *C*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1025D+04	0.1032D+04	0.1089D+04	0.1117D+04	0.1151D+04	0.1163D+04
ave 1	0.1211D+04	0.1239D+04	0.1244D+04	0.1287D+04	0.1351D+04	0.1302D+04
ave 2	0.8384D+03	0.82554+03	0.9336D+03	0.9469D+03	0.9514D+03	0.1023D+04
std dev	0.3079D+03	0.3859D+03	0.4395D+03	0.4884D+03	0.5511D+03	0.5454D+03
std dev 1	0.2796D+03	0.3465D+03	0.5096D+03	0.5524D+03	0.6167D+03	0.6030D+03
std dev 2	0.2053D+03	0.3037D+03	0.2803D+03	0.3380D+03	0.3839D+03	0.4384D+03

Table 8.11f. Chaboche law isotropic coefficient, Q, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1581D+05	0.1489D+05	0.1400D+05	0.1386D+05	0.1384D+05	0.1388D+05
ave 1	0.1593D+05	0.1517D+05	0.1419D+05	0.1397D+05	0.1404D+05	0.1404D+05
ave 2	0.1569D+05	0.1460D+05	0.1380D+05	0.1375D+05	0.1364D+05	0.1371D+05
std dev	0.1331D+04	0.1495D+04	0.2252D+04	0.2479D+04	0.2691D+04	0.2692D+04
std dev 1	0.1285D+04	0.1408D+04	0.1980D+04	0.2096D+04	0.2352D+04	0.2412D+04
std dev 2	0.1365D+04	0.1524D+04	0.2478D+04	0.2807D+04	0.2979D+04	0.2937D+04

Table 8.12a. Chaboche law modulus of elasticity, E, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9866D+07	0.9845D+07	0.9859D+07	0.9914D+07	0.9960D+07	0.9977D+07
ave 1	0.9540D+07	0.9556D+07	0.9602D+07	0.9617D+07	0.9670D+07	0.9705D+07
ave 2	0.1019D+08	0.1013D+08	0.1012D+08	0.1021D+08	0.1025D+08	0.1025D+08
std dev	0.4480D+06	0.4254D+06	0.4170D+06	0.4407D+06	0.4215D+06	0.4269D+06
std dev 1	0.2641D+06	0.2121D+06	0.2861D+06	0.2836D+06	0.2716D+06	0.2947D+06
std dev 2	0.3461D+06	0.3871D+06	0.3661D+06	0.3625D+06	0.3367D+06	0.3601D+06

Table 8.12b. Chaboche law yield strain, ε_y, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2315D-02	0.2296D-02	0.2271D-02	0.2236D-02	0.2173D-02	0.2142D-02
ave 1	0.2443D-02	0.2415D-02	0.2355D-02	0.2323D-02	0.2273D-02	0.2221D-02
ave 2	0.2188D-02	0.2176D-02	0.2188D-02	0.2149D-02	0.2072D-02	0.2062D-02
std dev	0.2581D-03	0.2149D-03	0.1866D-03	0.2021D-03	0.1913D-03	0.2201D-03
std dev 1	0.2835D-03	0.2117D-03	0.1855D-03	0.1794D-03	0.1615D-03	0.2298D-03
std dev 2	0.1426D-03	0.1380D-03	0.1455D-03	0.1855D-03	0.1641D-03	0.1773D-03

Table 8.12c. Chaboche law kinematic coefficient, a, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.9984D+04	0.1035D+05	0.1111D+05	0.1250D+05	0.1222D+05	0.1327D+05
ave 1	0.9545D+04	0.1000D+05	0.1133D+05	0.1237D+05	0.1171D+05	0.1289D+05
ave 2	0.1042D+05	0.1071D+05	0.1089D+05	0.1262D+05	0.1273D+05	0.1365D+05
std dev	0.1432D+04	0.1285D+04	0.1658D+04	0.1901D+04	0.1455D+04	0.1695D+04
std dev 1	0.1500D+04	0.1416D+04	0.2133D+04	0.2040D+04	0.1493D+04	0.1043D+04
std dev 2	0.1210D+04	0.1023D+04	0.9216D+03	0.1742D+04	0.1215D+04	0.2092D+04

Table 8.12d. Chaboche law isotropic exponent, b, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2668D+01	0.4065D+01	0.4588D+01	0.4607D+01	0.4442D+01	0.4243D+01
ave 1	0.3683D+01	0.5161D+01	0.5716D+01	0.5874D+01	0.5832D+01	0.5579D+01
ave 2	0.1652D+01	0.2969D+01	0.3460D+01	0.3339D+01	0.3052D+01	0.2907D+01
std dev	0.2565D+01	0.2549D+01	0.2209D+01	0.2129D+01	0.2288D+01	0.2716D+01
std dev 1	0.3128D+01	0.2959D+01	0.2510D+01	0.2140D+01	0.2331D+01	0.3082D+01
std dev 2	0.1145D+01	0.1354D+01	0.9584D+00	0.1128D+01	0.1082D+01	0.1300D+01

Table 8.12e. Chaboche law kinematic exponent, C, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.8522D+03	0.8002D+03	0.7341D+03	0.6307D+03	0.6527D+03	0.5696D+03
ave 1	0.9407D+03	0.8477D+03	0.7844D+03	0.6420D+03	0.7138D+03	0.6573D+03
ave 2	0.7638D+03	0.7527D+03	0.6838D+03	0.6193D+03	0.5916D+03	0.4819D+03
std dev	0.3247D+03	0.2326D+03	0.2005D+03	0.2078D+03	0.1848D+03	0.2130D+03
std dev 1	0.4045D+03	0.2634D+03	0.2371D+03	0.1860D+03	0.1779D+03	0.2022D+03
std dev 2	0.1779D+03	0.1854D+03	0.1382D+03	0.2270D+03	0.1708D+03	0.1856D+03

Table 8.12f. Chaboche law isotropic coefficient, Q, for random amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1189D+05	0.1309D+05	0.1392D+05	0.1394D+05	0.1411D+05	0.1384D+05
ave 1	0.1170D+05	0.1270D+05	0.1378D+05	0.1384D+05	0.1365D+05	0.1323D+05
ave 2	0.1208D+05	0.1348D+05	0.1406D+05	0.1405D+05	0.1457D+05	0.1445D+05
std dev	0.3559D+04	0.3652D+04	0.3377D+04	0.3412D+04	0.3550D+04	0.3710D+04
std dev 1	0.2658D+04	0.3071D+04	0.3090D+04	0.3124D+04	0.3226D+04	0.3306D+04
std dev 2	0.4265D+04	0.4116D+04	0.3636D+04	0.3675D+04	0.3793D+04	0.3980D+04

Table 8.13a. Chaboche law significance of the influence of various factors on the modulus of elasticity, E.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	-	*	-	-	-	-
M	*	*	*	*	*	*
HxM	*	*	-	-	-	-
L	-	-	-	-	-	-
HxL	-	-	-	-	-	-
MxL	-	-	-	-	-	-
HxMxL	-	*	-	*	*	*

Table 8.13b. Chaboche law significance of the influence of various factors on the yield strain, ε_y.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	-
M	*	*	*	*	*	-
HxM	*	*	*	*	*	*
L	*	*	*	*	*	*
HxL	*	*	*	*	*	*
MxL	-	-	-	-	-	-
HxMxL	-	-	*	*	*	*

Table 8.13c. Chaboche law significance of the influence of various factors on the kinematic coefficient, *a*.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	-	-	-	*	-	-
M	-	-	*	-	-	*
HxM	*	*	*	*	*	*
L	*	*	*	*	-	-
HxL	*	*	*	-	-	-
MxL	-	-	-	-	-	-
HxMxL	-	-	-	-	-	-

Table 8.13d. Chaboche law significance of the influence of various factors on the isotropic exponent, *b*.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	-	*	*	*	*
M	*	*	*	*	*	*
HxM	-	-	*	*	*	*
L	-	-	-	-	-	-
HxL	-	-	-	-	-	-
MxL	-	-	*	*	*	*
HxMxL	-	-	-	-	-	-

Table 8.13e. Chaboche law significance of the influence of various factors on the kinematic exponent, *C*.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	*	*	*	*
M	*	*	*	*	*	*
HxM	*	*	-	*	-	-
L	*	*	*	*	*	*
HxL	*	-	*	-	*	*
MxL	*	*	*	-	*	*
HxMxL	-	*	*	*	*	*

Table 8.13f. Chaboche law significance of the influence of various factors on the isotropic coefficient, *Q*.

Window	0-10	0-20	0-50	0-100	0-200	0-500
H	*	*	-	-	-	-
M	-	-	-	-	-	-
HxM	-	-	-	-	-	-
L	-	*	*	*	-	-
HxL	-	-	*	*	-	-
MxL	-	-	-	-	-	-
HxMxL	-	-	-	-	-	-

Table 8.14. Chaboche law significance of the dependence of the parameters on the window.

Parameter	a	b	C	E	ε_y	Q
W	*	*	-	-	*	-
WxH	*	*	*	*	-	*
WxM	-	-	-	*	-	-
WxHxM	-	-	-	-	-	-
WxL	*	-	*	*	-	-
WxLxH	-	-	*	-	*	-
WxLxM	-	-	-	-	-	-
WxLxHxM	-	-	*	-	-	-

Table 8.15. Chaboche law correlation matrix for window six.

5086	E	ε_y	a	b	C	Q
E	1.000	-0.509	0.315	-0.019	0.045	0.241
ε_y	-0.509	1.000	-0.339	0.099	-0.512	-0.282
a	0.315	-0.339	1.000	-0.302	-0.073	0.072
b	-0.019	0.099	-0.302	1.000	-0.463	0.424
C	0.045	-0.512	-0.073	-0.463	1.000	0.130
Q	0.241	-0.282	0.072	0.424	0.130	1.000

5454	E	ε_y	a	b	C	Q
E	1.000	-0.518	0.122	0.451	-0.167	0.178
ε_y	-0.518	1.000	-0.378	-0.264	-0.312	-0.140
a	0.122	-0.378	1.000	0.564	-0.503	-0.179
b	0.457	-0.264	0.564	1.000	-0.622	-0.322
C	-0.167	-0.312	-0.503	-0.622	1.000	0.251
Q	0.178	-0.140	0.178	-0.140	-0.179	1.000

5086 and 5454	E	ε_y	a	b	C	Q
E	1.000	-0.483	0.055	-0.044	-0.137	0.215
ε_y	-0.483	1.000	-0.314	0.021	-0.412	-0.215
a	0.055	-0.314	1.000	0.120	-0.211	-0.100
b	-0.044	0.021	0.120	1.000	-0.388	-0.355
C	-0.137	-0.412	-0.211	-0.388	1.000	0.164
Q	0.215	-0.215	-0.100	-0.355	0.164	1.000

Table 8.16a. B-L law modulus of elasticity, *E*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1002D+08	0.9998D+07	0.1013D+08	0.1014D+08	0.1016D+08	0.1012D+08
ave 1	0.9800D+07	0.9854D+07	0.1004D+08	0.1010D+08	0.1015D+08	0.1010D+08
ave 2	0.1024D+08	0.1014D+08	0.1023D+08	0.1019D+08	0.1017D+08	0.1014D+08
std dev	0.3743D+06	0.3507D+06	0.4883D+06	0.4535D+06	0.4246D+06	0.4288D+06
std dev 1	0.3546D+06	0.3637D+06	0.5000D+06	0.5098D+06	0.4911D+06	0.4681D+06
std dev 2	0.2410D+06	0.2692D+06	0.4579D+06	0.3838D+06	0.3454D+06	0.3848D+06

Table 8.16b. B-L law yield strain, ε_y, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.2328D-02	0.2294D-02	0.2174D-02	0.2134D-02	0.2101D-02	0.2093D-02
ave 1	0.2381D-02	0.2282D-02	0.2111D-02	0.2047D-02	0.1986D-02	0.1970D-02
ave 2	0.2275D-02	0.2306D-02	0.2236D-02	0.2221D-02	0.2217D-02	0.2217D-02
std dev	0.1279D-03	0.1622D-03	0.2758D-03	0.3094D-03	0.3185D-03	0.3316D-03
std dev 1	0.1194D-03	0.1991D-03	0.3197D-03	0.3706D-03	0.3719D-03	0.3628D-03
std dev 2	0.2410D+06	0.2692D+06	0.4579D+06	0.3838D+06	0.3454D+06	0.3848D+06

Table 8.16c. B-L law kinematic coefficient, *a*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.7315D+04	0.7716D+04	0.8982D+04	0.9380D+04	0.9820D+04	0.9954D+04
ave 1	0.7312D+04	0.8161D+04	0.9809D+04	0.1053D+05	0.1110D+05	0.1122D+05
ave 2	0.7318D+04	0.7272D+04	0.8154D+04	0.8227D+04	0.8541D+04	0.8686D+04
std dev	0.1251D+04	0.1304D+04	0.1925D+04	0.2157D+04	0.2126D+04	0.2089D+04
std dev 1	0.1375D+04	0.1430D+04	0.1915D+04	0.2228D+04	0.2164D+04	0.1994D+04
std dev 2	0.1114D+04	0.1020D+04	0.1541D+04	0.1298D+04	0.1039D+04	0.1240D+04

Table 8.16d. B-L law isotropic exponent, *b*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.6173D+01	0.5073D+01	0.3543D+01	0.2976D+01	0.2758D+01	0.2698D+01
ave 1	0.6703D+01	0.5650D+01	0.4492D+01	0.3935D+01	0.3808D+01	0.3797D+01
ave 2	0.5644D+01	0.4496D+01	0.2595D+01	0.2016D+01	0.1707D+01	0.1598D+01
std dev	0.1133D+01	0.1332D+01	0.1795D+01	0.2005D+01	0.2168D+01	0.2285D+01
std dev 1	0.9834D+00	0.1144D+01	0.1790D+01	0.2056D+01	0.2178D+01	0.2249D+01
std dev 2	0.1019D+01	0.1255D+01	0.1200D+01	0.1404D+01	0.1564D+01	0.1723D+01

Table 8.16e. B-L law kinematic exponent, *C*, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1177D+04	0.1261D+04	0.1315D+04	0.1404D+04	0.1417D+04	0.1479D+04
ave 1	0.1410D+04	0.1542D+04	0.1569D+04	0.1655D+04	0.1678D+04	0.1769D+04
ave 2	0.9442D+03	0.9814D+03	0.1061D+04	0.1152D+04	0.1156D+04	0.1190D+04
std dev	0.3491D+03	0.4413D+03	0.4664D+03	0.4900D+03	0.5207D+03	0.5654D+03
std dev 1	0.3301D+03	0.4122D+03	0.4161D+03	0.4503D+03	0.4596D+03	0.5055D+03
std dev 2	0.1623D+03	0.2505D+03	0.3647D+03	0.3881D+03	0.4414D+03	0.4655D+03

Table 8.16f. B-L law isotropic coefficient, Q, for constant amplitude strain history.

Window	0-10	0-20	0-50	0-100	0-200	0-500
ave	0.1276D+05	0.1200D+05	0.1150D+05	0.1135D+05	0.1099D+05	0.1085D+05
ave 1	0.1328D+05	0.1270D+05	0.1196D+05	0.1158D+05	0.1111D+05	0.1104D+05
ave 2	0.1223D+05	0.1130D+05	0.1103D+05	0.1111D+05	0.1086D+05	0.1066D+05
std dev	0.1138D+04	0.1307D+04	0.1727D+04	0.1950D+04	0.1771D+04	0.1704D+04
std dev 1	0.8913D+03	0.9662D+03	0.1503D+04	0.1753D+04	0.1530D+04	0.1449D+04
std dev 2	0.1118D+04	0.1227D+04	0.1809D+04	0.2104D+04	0.1975D+04	0.1907D+04

Table 8.17a. B-L law significance of the influence of various factors on the modulus of elasticity, E.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	*	*	-	-	-	-
L	-	-	-	-	-	-
MxL	-	-	-	-	-	-

Table 8.17b. B-L law significance of the influence of various factors on the yield strain, ε_y.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	-	-	*	*	*	*
L	*	*	*	*	*	*
MxL	-	-	-	-	-	-

Table 8.17c. B-L law significance of the influence of various factors on the kinematic coefficient, a.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	-	-	*	*	*	*
L	-	-	*	*	*	*
MxL	-	-	-	-	-	-

Table 8.17d. B-L law significance of the influence of various factors on the isotropic exponent, b.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	*	*	*	*	*	*
L	*	*	-	-	*	*
MxL	-	-	-	-	-	-

Table 8.17e. B-L law significance of the influence of various factors on the kinematic coefficient, C.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	*	*	*	*	*	*
L	*	*	*	*	*	*
MxL	-	-	-	-	-	-

Table 8.17f. B-L law significance of the influence of various factors on the isotropic coefficient, Q.

Window	0-10	0-20	0-50	0-100	0-200	0-500
M	*	*	-	-	-	-
L	-	-	-	*	*	*
MxL	-	-	-	-	-	-

Table 8.18. B-L law significance of the dependence of the parameters on the window.

Parameter	E	ε_y	a	b	C	Q
W	*	*	*	*	*	*
WxM	*	*	*	*	*	-
WxL	*	*	*	-	*	-
WxMxL	-	-	-	-	-	-

Table 8.19. B-L law correlation matrix for window six.

	5086					
	E	ε_y	a	b	C	Q
E	1.000	-0.755	0.496	-0.478	0.366	0.139
ε_y	-0.755	1.000	-0.887	-0.431	-0.558	0.076
a	0.496	-0.887	1.000	0.272	0.301	0.003
b	0.478	-0.431	0.272	1.000	-0.002	-0.659
C	0.366	-0.558	0.301	-0.002	1.000	0.065
Q	0.139	0.076	0.003	-0.659	0.065	1.000

	5454					
	E	ε_y	a	b	C	Q
E	1.000	-0.769	0.584	0.121	0.336	0.249
ε_y	-0.769	1.000	-0.712	-0.390	-0.669	0.037
a	0.584	-0.712	1.000	-0.015	0.069	0.313
b	0.121	-0.390	-0.015	1.000	0.395	-0.832
C	0.336	-0.669	0.069	0.395	1.000	-0.246
Q	0.249	0.037	0.313	-0.832	-0.246	1.000

	5086 and 5454					
	E	ε_y	a	b	C	Q
E	1.000	-0.686	0.394	0.279	0.283	0.185
ε_y	-0.686	1.000	-0.842	-0.517	-0.663	-0.0098
a	0.394	-0.842	1.000	0.416	0.455	0.172
b	0.279	-0.517	0.416	1.000	0.367	-0.574
C	0.285	-0.663	0.455	0.367	1.000	-0.033
Q	0.185	0.0098	0.172	-0.514	-0.033	1.000

Table 8.20. Comparison of the values of the modulus of elasticity (Msi).

		Kinematic		Isotropic		Chaboche		B-L	
		0-10	0-500	0-10	0-500	0-10	0-500	0-10	0-500
Cyclic	ave	8.868	9.252	8.593	9.022	10.01	9.890	10.02	10.12
	ave 1	8.828	9.201	8.530	8.936	9.857	9.813	9.800	10.10
	ave 2	8.907	9.206	8.656	9.108	10.16	9.967	10.24	10.14
Random	ave	9.228	9.753	9.022	9.817	9.866	9.977		
	ave 1	9.017	9.485	8.805	9.575	9.540	9.705		
	ave 2	9.438	10.02	9.238	10.06	10.19	10.25		

9. THE PROBLEM OF INITIAL CONDITIONS

Computations in the previous section were made under the assumption that the material is "virgin" or in a standard state. This assumption was used to justify the following standard initial conditions: a) The kinematic law, initial condition $\alpha(0) = 0$, with parameters E, E_p, σ_y. b) The isotropic law, initial condition $\alpha(0) = 0$, with parameters E, E_p, σ_y. c) The Chaboche law, initial condition $\chi(0) = R(0) = 0$, $\varepsilon_h = -\varepsilon_\ell = \varepsilon_y$, with parameters a, b, C, Q, ε_y, E. d) The B-L law, initial condition $\alpha(0) = \beta(0) = 0$, with parameters a, b, C, Q, ε_y, E.

As was stated for windows 1-6 the standard initial condition were used for all four laws. Standard initial conditions for windows 7-11 were used for the kinematic and isotropic laws together with the initial value of stress from experimental data at the beginning of the window. The standard initial condition is, of course, not correct because the material already has experienced a strain history in the case of windows 7-11.. Nevertheless, for the crude kinematic and isotropic models standard initial conditions were used. For the Chaboche and B-L law standard initial conditions for windows 7-11 were not used because effects on error Θ are more significant. The error and optimal parameter data are reported for two replica experiments for material 1 (5086), with a mean strain level of 0.000 and for windows 1 and 7 in the following tables.

Each of the four laws are analyzed to determine the effect of initial conditions on the error in windows one and seven. The data for window one of the kinematic law is reported in Table 9.1a for the case of standard initial conditions $\alpha(0) = 0$, and for the case of optimal initial conditions. For window seven the data are reported in Table 9.1b for the following cases: standard initial conditions, $\alpha(0) = 0$; initial conditions for window seven taken at the end of window 1, when $\alpha(0) = 0$ is assumed at the beginning of window one; and optimal initial conditions for window seven determined by minimizing the error over window seven for both the parameter values and initial conditions. This last case results in the least error as it should. Estimates of the constitutive law parameters obtained from published data (see Appendix II) are used to compute the error in Table 9.2c for the cases of window one with standard initial conditions, window seven with initial conditions determined from window one, and window seven with standard initial conditions. For comparison purposes, the error of reproducibility of the samples (σ_1-σ_2 error) is reported in Table 9.1d. Error for each of the eleven windows is reported in Table 9.1e for the case of standard initial conditions with the initial stress taken as the

measured stress at the beginning of the window. Larger errors are reported in Table 9.1f when the optimal parameter values and optimal initial conditions are computed in window one and used in windows one through six. Comparing Tables 9.1b, 9.1e and 9.1f shows that selecting optimal initial conditions and parameters from window one and using these values in other windows results in larger errors than the case of optimal selection for each window.

Analogous tables are reported for the isotropic law. Recall that the number of internal variables, parameters, and initial conditions are the same for the isotropic and kinematic law. In Table 9.2b are reported the data for the case when optimal parameters and initial conditions for window one are used in window seven. Once more, as in the case of the kinematic law, we see that the isotropic law does not perform as well when optimal parameters and initial values are taken from window one and subsequently used for other windows.

Analogous data for the Chaboche law are reported in Tables 9.3a-d. Note that there are four internal variables with initial conditions and six parameters in this law. If all are determined optimally, ten coefficients must be estimated. The values of Θ in windows 1-6 are reported in Table 9.3e when optimal values of the parameters and initial conditions from window one are used. Comparing Tables 9.3c, 9.3d and 9.3e we see that determining the parameters values and initial conditions from the initial window, window one, leads to larger errors than when these values are computed separately for each window using standard initial conditions. Data for the B-L law are reported in analogous format.

From the reported data we can conclude that, in general, initial conditions do not influence performance of the laws except during first few cycles of a given history. This illustrates the well known effect of fading memory. Because of this effect our analysis is limited to the case of standard initial conditions especially in the first six windows. From the data presented in Tables 7.1a-c–7.10a-c, 9.1f, 9.2e, 9.3e, 9.4d and from the results of Section 8 the use of the optimal parameters and initial conditions from window one (0-10) for window six (0-500) leads to higher error than the use of optimal parameters and standard initial condition determined and used in window six. This agrees with the data in Section 8 which indicate that parameters are window dependent, i.e., they are not sufficiently "constant" with respect to the windows which means that the models are not sufficiently adequate. Tables 9.1f, 9.2e, 9.3e and 9.4d also show that performance of the

laws deteriorates with increasing window length, which means that the location of maximal error is at the end of the considered window.

Table 9.1a. Kinematic law. The error Θ for window 1 and optimal parameters .

Sample	E	E_p	σ_y	α	Θ	Remark
1	0.9571 (7)	0.2918 (7)	0.2546 (5)	0	12.93	Standard initial
2	0.9868 (7)	0.2814 (7)	0.2600 (5)	0	13.50	conditions
1	0.9759 (7)	0.2926 (7)	0.2644 (5)	0.9783	9.54	Optimal initial
2	1.0226 (7)	0.3086 (7)	0.2615 (5)	0.9590	10.06	conditions

Table 9.1b. Kinematic law. The error Θ for window 7 and optimal parameters .

Sample	E	E_p	σ_y	α	Θ	Remark
1	0.8956 (7)	0.2138 (7)	0.3079 (5)	0	11.06	Standard initial
2	0.9261 (7)	0.2098 (7)	0.3099 (5)	0	11.21	conditions
1	0.9889 (7)	0.3847 (7)	0.2778 (5)	-0.3301 (1)	11.06	Initial conditions
2	1.0205 (7)	0.3773 (7)	0.2793 (5)	-0.3435 (1)	6.72	from window 1
1	0.9764 (7)	0.3620 (7)	0.2838 (5)	-0.2946 (1)	6.17	Optimal initial
2	1.0027 (7)	0.3402 (7)	0.2892 (5)	-0.2839 (1)	6.44	conditions

Table 9.1c. Kinematic law. The error Θ for window 7 and handbook values of parameters .

Sample	Window	E	E_p	σ_y	α	Θ	Remark
1	0-10	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	34.45	Standard initial
2	0-10	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	32.60	conditions
1	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	-0.4074	30.89	Initial conditions
2	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	-0.2936 (-2)	28.86	from window 1
1	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	31.45	Standard initial
2	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	28.90	conditions

Table 9.1d. Error Θ for reproducibility.

Window	Θ
0-10	4.39
10-20	4.90

Table 9.1e. Kinematic law. The error Θ for windows 1-11, standard initial condition, and optimal parameters for every window.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	12.93	13.38	13.86	14.63	15.64	17.23
2	13.50	14.23	14.98	15.56	16.25	17.47

Sample	0-10	10-20	20-50	50-100	100-200	200-500
1	12.93	11.06	11.31	10.88	10.60	10.71
2	13.50	11.21	11.37	10.68	10.43	10.25

Table 9.1f. Kinematic law. The error Θ for windows 1-6, optimal parameters, and initial condition determined in window 1.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	9.54	12.73	16.06	18.54	20.82	24.49
2	10.06	12.57	15.57	18.02	21.16	24.72

Table 9.2a. Isotropic law. The error Θ optimal and parameters for window 1.

Sample	E	E_p	σ_y	α	Θ	Remark
1	0.8500 (7)	0.1631 (3)	0.2947 (5)	0	16.65	Standard initial
2	0.8766 (7)	0.3268 (1)	0.2976 (5)	0	17.31	conditions
1	0.8500 (7)	0.1532 (3)	0.2047 (5)	-0.2525 (-3)	16.64	Optimal initial
2	0.8766 (7)	0.1405 (1)	0.2976 (5)	-0.1057 (-3)	17.31	conditions

Table 9.2b. Isotropic law. The error Θ and optimal parameters for window 7.

Sample	E	E_p	σ_y	α	Θ	Remark
1	0.8699 (7)	0.7379 (6)	0.3035 (5)	0	15.44	Standard initial
2	0.8700 (7)	0.7099 (6)	0.3034 (5)	0	15.64	conditions
1	0.8699 (7)	0.7378 (6)	0.3037 (5)	-0.1929 (-1)	15.44	Initial conditions
2	0.8924 (7)	0.7097 (6)	0.3034 (5)	-0.2936 (-2)	15.64	from window 1
1	0.8700(7)	0.4289 (6)	0.3210 (5)	+0.4009 (-1)	15.44	Optimal initial
2	0.8924 (7)	0.7097 (6)	0.3034 (5)	-0.2839 (-3)	15.64	conditions

Table 9.2c. Isotropic law. The error Θ and handbook values of the parameters.

Sample	Window	E	E_p	σ_y	α	Θ	Remark
1	0-10	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	34.85	Standard initial
2	0-10	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	32.99	conditions
1	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	-0.1929 (1)	31.35	Initial conditions
2	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	-0.2956 (-2)	29.33	from window 1
1	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	31.35	Standard initial
2	10-20	1.0400 (7)	0.1040 (6)	0.3857 (5)	0	29.33	conditions

Table 9.2d. Isotropic law. The error Θ for windows 1-11, standard initial conditions, and optimal parameters for every window.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	16.65	17.36	17.92	18.54	19.09	19.68
2	17.31	17.69	18.42	18.95	19.63	20.47

Sample	0-10	10-20	20-50	50-100	100-200	200-500
1	16.65	15.07	14.21	12.98	12.84	12.08
2	17.31	15.64	15.04	13.73	12.58	11.45

Table 9.2e. Isotropic law. The error Θ for windows 1-6, optimal parameters and initial condition determined in window 1.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	16.65	19.84	22.67	25.64	28.37	30.53
2	17.31	20.14	23.06	25.55	26.99	33.30

Table 9.3a. Chaboche law. The error Θ and optimal parameters for window 1.

Sample	a	b	C	E	Q
1	0.9805 (4)	0.5534 (1)	0.1172 (4)	1.0096 (7)	0.1705 (5)
2	0.9121 (4)	0.6074 (1)	0.1225 (4)	1.0254 (7)	0.1616 (5)
1	1.1423 (4)	0.5404 (1)	0.1382 (4)	0.9675 (7)	0.1987 (5)
2	1.0763 (4)	0.5232 (1)	0.1371 (4)	0.9987 (7)	0.1628 (5)

Sample	ε_h	ε_ℓ	χ	R	Θ	Remarks
1	0.2237 (-2)	-0.2237 (-2)	0	0	9.47	Standard initial
2	0.2232 (-2)	-0.2232 (-2)	0	0	9.54	conditions
1	0.2954 (-2)	-0.5686 (-3)	0.8930 (4)	0.1463 (4)	3.73	Optimal initial
2	0.2769 (-2)	-0.7983 (-3)	0.6469 (4)	0.2137 (4)	4.29	conditions

Table 9.3b. Chaboche law. The error Θ and optimal parameters for window 7.

Sample	a	b	C	E	Q	ε_h
1	0.6251 (4)	1.7539	0.1785 (3)	0.7926 (7)	0.2179 (5)	0.3717 (-2)
2	0.9105 (4)	0.8771	0.6117 (3)	0.7966 (7)	0.1854 (5)	0.2754 (-2)
1	0.1697 (5)	0.1306 (1)	0.1230 (4)	0.9981 (7)	0.2157 (5)	-0.5140 (-3)
2	0.1560 (5)	0.1588 (1)	0.1199 (4)	1.0228 (7)	0.2002 (5)	-0.4721 (-3)
1	0.1713 (5)	0.1398 (1)	0.1256 (4)	0.9982 (7)	0.2163 (5)	-0.5287 (-3)
2	0.1564 (5)	0.1593 (1)	0.1197 (4)	1.0228 (7)	0.2003 (5)	-0.4723 (-3)

Sample	ε_ℓ	χ	R	Θ	Remarks
1	-0.3717 (-2)	0	0	24.32	Standard initial
2	-0.2754 (-2)	0	0	26.49	conditions
1	-0.3150 (-2)	-0.9052 (4)	0.5102 (4)	3.16	Optimal initial
2	-0.3270(-2)	0.8532 (4)	0.5409 (4)	3.01	conditions from window 1
1	-0.3099 (-3)	-0.8893 (4)	0.5124 (4)	3.15	Optimal initial
2	-0.3260 (-2)	-0.8532 (4)	0.5409 (4)	3.00	conditions

Table 9.3c. Chaboche law. The error Θ and handbook values of the parameters.

Sample	Window	a	b	C	E	Q
1	0-10	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)
2	0-10	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)
1	10-20	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)
2	10-20	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)
1	10-20	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)
2	10-20	0.6000 (4)	1.0000	0.4000 (4)	1.04 (7)	0.1000 (5)

Sample	ε_h	ε_ℓ	χ	R	Θ	Remarks
1	0.2596 (-2)	-0.2596 (-2)	0	0	24.25	Standard initial
2	0.2596 (-2)	-0.2596 (-2)	0	0	22.30	conditions
1	0.2596 (-2)	-0.2596 (-2)	-0.5999 (4)	0.7421 (3)	63.17	Initial conditions
2	0.2596 (-2)	-0.2596 (-2)	-0.5999 (4)	0.7427 (3)	61.10	from window 1
1	0.2596 (-2)	-0.2596 (-2)	0	0	55.62	Standard initial
2	0.2596 (-2)	-0.2596 (-2)	0	0	53.37	conditions

Table 9.3d. Chaboche law. The error Θ for windows 1-6, standard initial conditions, and optimal parameters for every window.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	9.47	9.26	9.20	9.16	9.26	9.34
2	9.54	9.28	9.20	9.18	8.91	8.83

Table 9.3e. Chaboche law. The error Θ for windows 1-6, optimal parameters and initial conditions determined in window 1.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	3.73	7.71	13.75	16.04	17.48	17.22
2	4.29	5.91	10.30	12.69	13.67	13.35

Table 9.4a. B-L law. The error Θ and optimal parameters for window 1.

Sample	*a*	*b*	*C*	*E*	*Q*
1	0.6977 (4)	0.5617 (1)	0.1412 (4)	0.9935 (7)	0.1252 (5)
2	0.7254 (4)	0.6118 (1)	0.1351 (4)	1.0309 (7)	0.1323 (5)
1	0.5675(4)	0.6674 (1)	0.1772 (4)	0.9424 (7)	0.1393 (5)
2	0.5522 (4)	0.6938 (1)	0.1788 (4)	0.9678 (7)	0.1577 (5)

Sample	ε_y	α	β	Θ	Remarks
1	0.2464 (-2)	0	0	8.13	Standard initial
2	0.2360 (-2)	0	0	7.98	conditions
1	0.2479 (-2)	0.4249 (1)	0.6308 (0)	6.64	Optimal initial
2	0.2285 (-2)	0.2401 (1)	4.979 (1)	6.97	conditions

Table 9.4b. B-L law. The error Θ and optimal parameters for window 2.

Sample	*a*	*b*	*C*	*E*	*Q*
1	0.8072 (4)	0.4968 (1)	0.1709 (4)	0.9558 (7)	0.1155 (5)
2	0.7920 (4)	0.3714 (1)	0.2085 (4)	0.9795 (7)	0.1305 (5)
1	0.8048 (4)	0.5532 (1)	0.1897 (4)	0.9590 (7)	0.1270 (5)
2	0.9340 (4)	0.4854 (1)	0.1724 (4)	1.0162 (7)	0.1428 (5)
1	0.8075 (4)	0.5167 (1)	0.1948 (4)	0.9606 (7)	0.1293 (5)
2	0.8147 (4)	0.4948 (1)	0.1952 (4)	0.9895 (7)	0.1452 (5)

Sample	ε_y	α	β	Θ	Remarks
1	0.2601 (-2)	0	0	6.17	Standard initial
2	0.2546 (-2)	0	0	6.37	conditions
1	0.2189 (-2)	-3.4373 (0)	0.1691 (2)	5.93	Initial conditions
2	0.1981 (-2)	3.5711 (0)	0.1877 (2)	6.05	from window 1
1	0.2071 (-2)	-0.3973 (0)	0.2354 (2)	5.83	Optimal initial
2	0.2038 (-2)	-2.4768 (0)	0.1976 (2)	5.98	conditions

Table 9.4c. B-L law. The error Θ for windows 1-6, standard initial conditions, and optimal parameters for every window.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	8.13	7.83	8.00	7.89	7.88	7.73
2	7.98	7.89	8.21	8.62	8.62	8.88

Table 9.4d. B-L law. The error Θ for windows 1-6, optimal parameters and initial conditions determined in window 1.

Sample	0-10	0-20	0-50	0-100	0-200	0-500
1	6.64	9.95	13.67	14.36	14.70	14.51
2	6.97	11.43	16.05	17.79	17.36	16.96

10. THE PROBLEM OF RELIABILITY OF COMPUTATIONAL ANALYSES

Let us address the most simple case of a typical one-dimensional quasistatic problem of plasticity. Let $I = [0,1]$ and $W = [0,T]$. Find the function $u(x,t)$ and the stresses $\sigma(x,t)$, $(x,t) \varepsilon I \times W$ such that

$$\frac{\partial \sigma}{\partial x}(x,t) = 0 \tag{10.1}$$

$$\frac{\partial u}{\partial x}(x,t) = \varepsilon(x,t) \tag{10.2}$$

$$u(x,0) = \sigma(x,0) = 0 \tag{10.3}$$

$$u(0,t) = 0 \tag{10.4}$$

and

Problem A

$$u(1,t) = h(t) \quad h(0) = 0 \tag{10.5a}$$

Problem B

$$\sigma(1,t) = g(t) \quad g(0) = 0 \tag{10.5b}$$

Denote the time derivatives

$$\dot{\sigma}(x,t) = \frac{\partial \sigma}{\partial t}(x,t)$$
$$\dot{\varepsilon}(x,t) = \frac{\partial \varepsilon}{\partial t}(x,t)$$

and assume that σ and ε are related by the constitutive law described in Section 6, in general and in particular by the four laws under consideration.

Obviously, Problem A is identical with the experiments described in this study. Hence the errors of the computational analyses are given in Tables 7.1-7.10. We can also assume, that in Problem B, the stress $\sigma(1,t)$ is taken from experimental measurements and strain is computed from the constitutive law under consideration. The results for window 1 (i.e., 0-10 cycles) are analyzed for cyclic load. For each case in Problem B the

optimal parameters are separately computed for the sample under consideration. The methods described in Section 7 are used for parameter identification. An error measure is defined as in Section 4.

$$\Theta = \frac{\max\limits_{w_{b,e}} \left| \left(\varepsilon_s - \varepsilon_{pred} \right) \right|}{\max\limits_{w_{b,e}} \frac{1}{2} \left| \left(\varepsilon_s + \varepsilon_{pred} \right) \right|}$$

where max is taken over the first 10 cycles, ε_s is measured (sample) strain, and ε_{pred} is computed (predicated) strain.

Selected results are reported for the Chaboche, isotropic and kinematic constitutive laws. The mean, standard deviation, two smallest, and two largest errors as well as the mean strain and material are reported in Table 10.1a for optimal parameters and the Chaboche law. The max error is not related to the mean strain level of the sample. In Table 10.1b are reported data for the case when parameters are determined from published data in handbooks and the literature. Analogous data for the isotropic law are given in Table 10.2a and b. For the isotropic law, handbook parameters lead to smaller mean error than the error for optimal coefficients computed for each sample. Finally we report the data for the kinematic law in Table 10.3a and 10.3b.

The optimal coefficients could also be computed from experimental data when stress is understood as the independent variable and strain as the dependent variable. Data are reported in Table 10.4 to illustrate this computation for the kinematic law. The value of Θ for the kinematic law is computed for predicted strain when the parameters are computed optimally in window one with stress understood as the independent variable. These values should be compared with the first column in Table 7.1a. As expected the reliability coefficient Θ is better for the predicted strain compared to Tables 10.3a and 10.3b. However, the optimal parameters determined in this way will perform worse for Problem A.

We have seen large uncertainties in the accuracy of constitutive laws. Hence major a question arises as how to formulate problems of plasticity in a manner which incorporates this uncertainty. It seems that the only practical way is to establish brackets in between which the true solution lies. Given a constitutive law, brackets for the parameters, and some information about the nature of possible history of the material, it is in principle possible to formulate a mathematical problem to find the guaranteed (and minimal)

brackets for the solution when input data ranges over the entire admissible set. This approach was analyzed in an idealized setting in [16].

Another more practical possibility is to utilize the covariance matrix and formulate sets of material properties by Monte Carlo methods to obtain the desired brackets. Few laws could be used because of uncertainties in selection of the laws. The main difficulty with this approach is that the covariance matrix is usually not available. When a large experimental material data base is available, a statistical assumption must be made regarding the form of the underlying distribution function, Gaussian or log Gaussian, etc. The Gaussian assumption can lead to negative values of some parameters which is physically impossible. The log Gaussian approach avoids this difficulty of negative parameter values but still could lead to physically impossible values. For example, there is a finite probability that $E_p > E$ for the kinematic law with a log Gaussian distribution function. On the other hand, utilizing the "worst" case approach will be too pessimistic.

As an illustration, let us consider a constant amplitude cyclic strain case with a mean strain of zero. Using a covariance matrix based on the log normal distribution we simulated 10 sets of sample parameters for every law and computed the reversal stress at tension and compression reversals. In Figures 10.1 a-d we report peak or envelope curves for the four constitutive laws, window six (0-500), and "virgin" (i.e., standard) initial conditions. Also plotted are representative curves for the second and fifth simulated sample. In Table 10.5, the error, Θ, is reported for 10 simulated samples and each constitutive law (i.e., comparison of the measured and simulated values). For isotropic hardening the simulation which results in $E_p > E$ has been excluded, and only seven simulations are reported in this case. The same bracketing could be performed for problem B when stress is prescribed. It is expected that the brackets would be further apart than those calculated for Problem A.

Table 10.1a. Chaboche law. Reliability of the strain computation for optimal sample parameters.

	Θ (%)	Mean Strain	Material
mean	76.65		
std dev	33.76		
min 1	33.90	0.000	5454
min 2	38.19	-0.002	5454
max 1	119.25	0.000	5086
max 2	118.81	0.000	5086

Table 10.1b. Chaboche law. Reliability of the strain computation for handbook parameters.

	Θ (%)	Mean Strain	Material
mean	133.72		
std dev	55.08		
min 1	34.66	0.002	5454
min 2	36.[illegible]2	-0.004	5086
max 1	190.00	0.000	5086
max 2	187.54	0.000	5086

Table 10.2a. Isotropic law. Reliability of the strain computation for optimal sample parameters.

	Θ (%)	Mean Strain	Material
mean	141.06		
std dev	15.95		
min 1	110.13	-0.002	5086
min 2	119.96	0.006	5454
max 1	172.96	0.000	5086
max 2	171.63	0.000	5086

Table 10.2b. Isotropic law. Reliability of the strain computation for handbook parameters.

	Θ (%)	Mean Strain	Material
mean	89.92		
std dev	19.44		
min 1	51.13	0.000	5086
min 2	52.89	0.000	5086
max 1	112.29	0.006	5454
max 2	112.29	-0.006	5454

Table 10.3a. Kinematic law. Reliability of the strain computation for optimal sample parameters.

	Θ (%)	Mean Strain	Material
mean	84.13		
std dev	10.98		
min 1	55.89	0.000	5454
min 2	62.08	-0.002	5454
max 1	101.38	0.006	5454
max 2	96.26	-0.004	5086

Table 10.3b. Kinematic law. Reliability of the strain computation for handbook parameters.

	Θ (%)	Mean Strain	Material
mean	89.93		
std dev	19.44		
min 1	51.13	0.000	5086
min 2	52.89	0.000	5086
max 1	112.29	0.006	5454
max 2	112.28	-0.006	5454

Table 10.4. Kinematic law. Reliability of the strain computation for optimal sample parameters computed for stress as the independent variable.

	Θ (%)	Mean Strain	Material
mean	64.69		
std dev	14.11		
min 1	40.79	-0.002	5454
min 2	43.10	-0.000	5086
max 1	83.15	0.006	5454
max 2	82.50	-0.006	5454

Table 10.5. The value of Θ of the accuracy of simulated samples.

Sample	1	2	3	4	5	6	7	8	9	10
Kinematic	33.36	36.20	37.18	27.56	20.12	21.40	30.65	32.73	24.06	25.10
Isotropic	33.26	43.22	37.89	31.27	29.13	57.05	33.77	-	-	-
Chaboche	11.43	12.68	13.13	14.51	18.55	11.68	13.44	22.92	12.01	13.83
B-L	14.43	15.17	17.44	10.60	16.91	17.75	15.61	11.06	.3.91	21.94

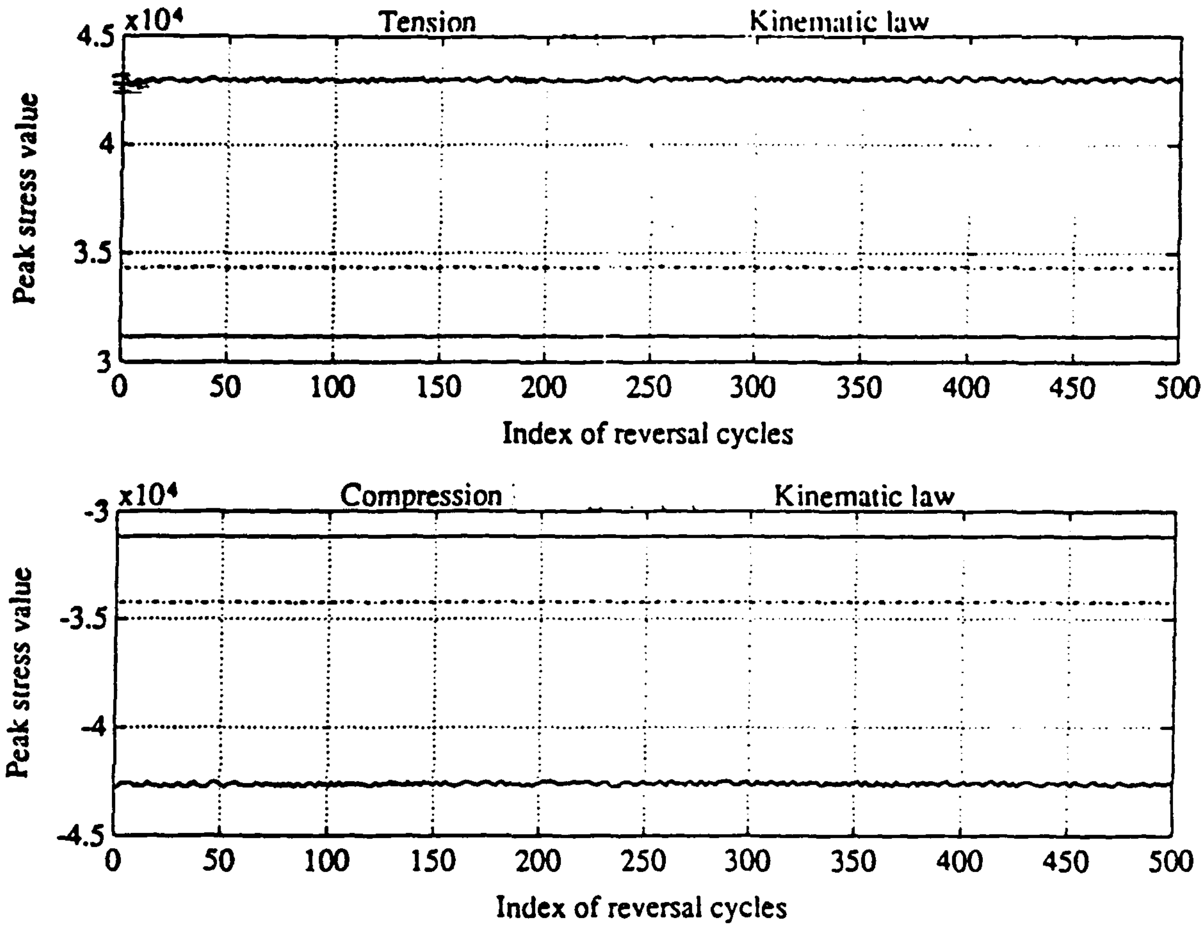

Figure 10.1a. Envelop of reversal stresses for ten sets of simulated parameters of the kinematic law (solid lines) and the reversal stress for the second and fifth set (dotted lines).

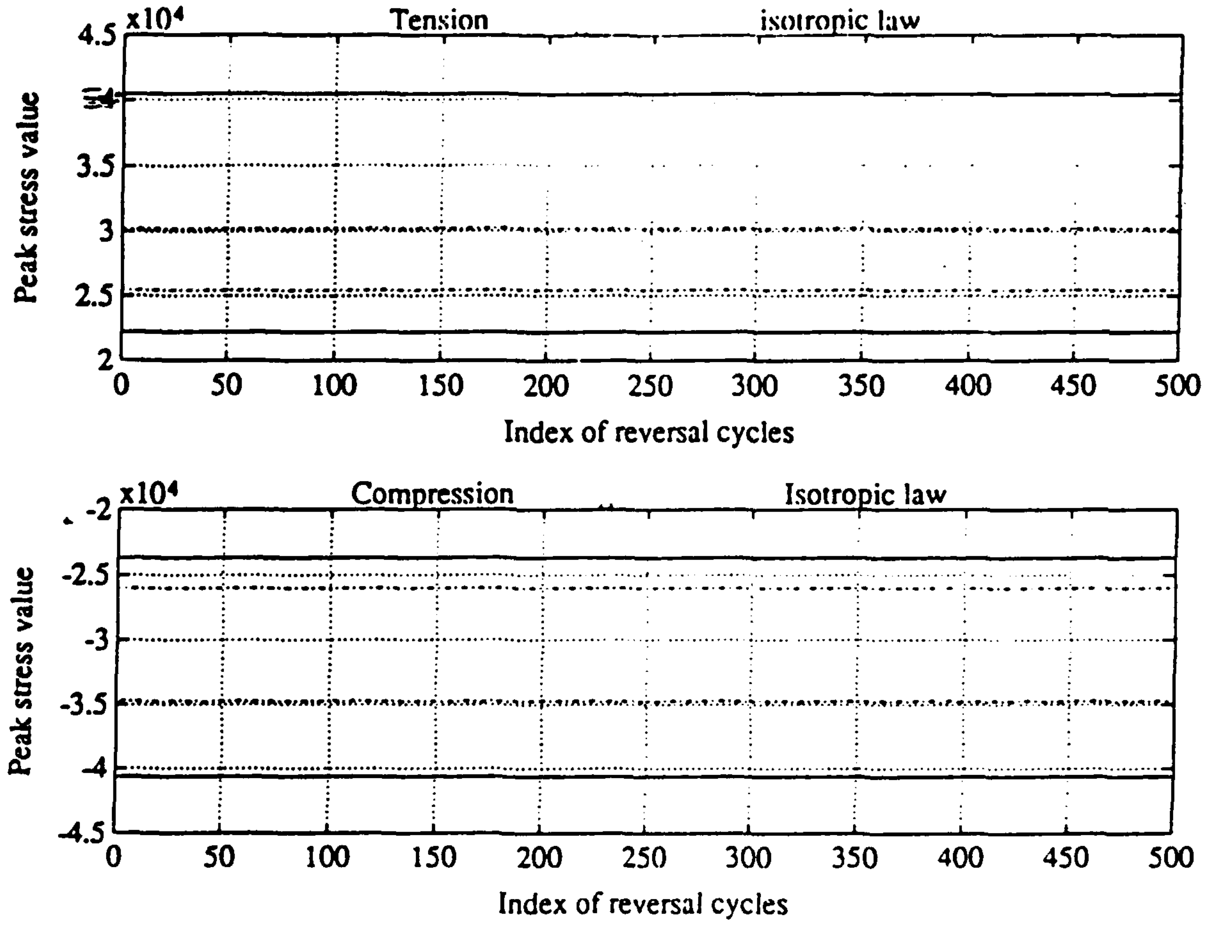

Figure 10.1b. Envelop of reversal stresses for ten sets of simulated parameters of the isotropic law (solid lines) and the reversal stress for the second and fifth set (dotted lines).

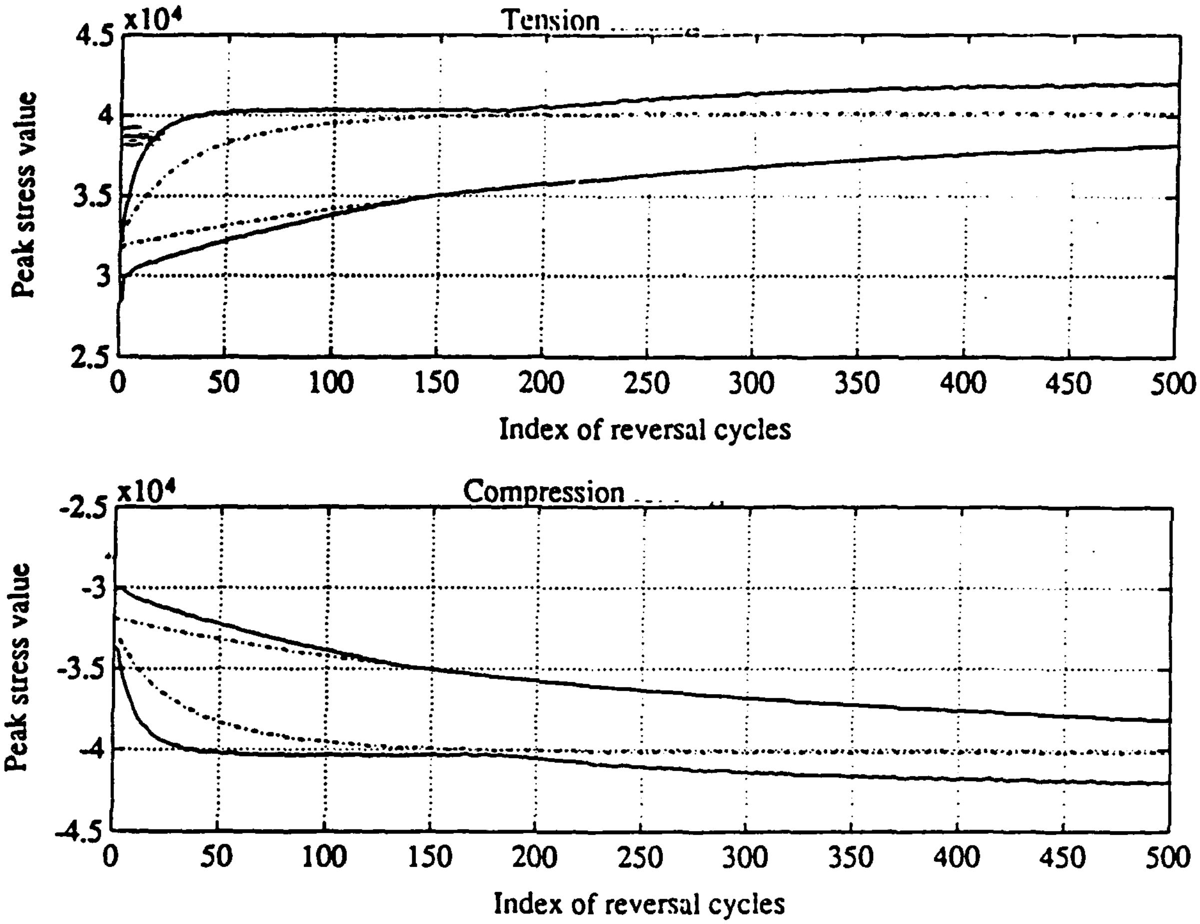

Figure 10.1c. Envelop of reversal stresses for ten sets of simulated parameters of the Chaboche law (solid lines) and the reversal stress for the second and fifth set (dotted lines).

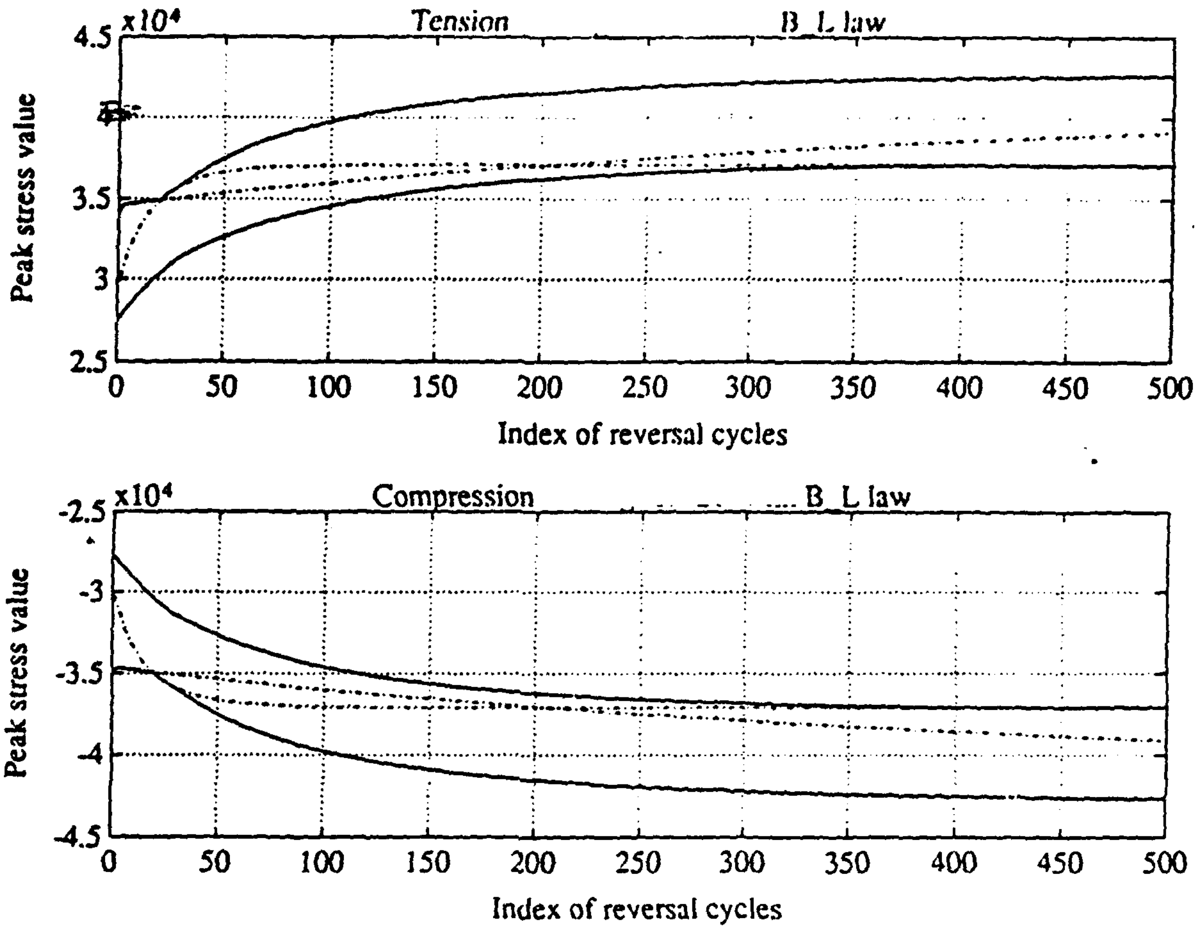

Figure 10.1d. Envelop of reversal stresses for ten sets of simulated parameters of the B-L law (solid lines) and the reversal stress for the second and fifth set (dotted lines).

11. CONCLUSIONS

In Sections 3 and 4 we have seen that the measured relation between strain and stress is completely reliable so that measurements error does not influence conclusions about material behavior. Based on this fact the presented data leads to the following conclusions.

a) Although the chemical composition of 5086 and 5454 aluminum is very similar, the material properties vary (e.g. variation of Rockwell B data). This variation results in a reproducibility error of the order of 5% with a maximum of the order of 9-10% for both materials. The reproducibility is not influenced by various factors such as history, mean strain level and material for histories ranging up to 200 cycles. After 200 cycles we see significant differences between the two materials.

b) All laws significantly increase the uncertainty characterized by reproducibility.

c) In general, use of standard initial conditions does not affect the quantitative performance of the laws.

d) The accuracy of the laws and values of parameters are influenced by windows and history. This indicates that the laws are inadequate because the parameters are not "constant." The criterion of independence of the parameters with respect to factors such as window and history could be used for assessment of the quality of a constitutive law.

e) Experiments performed with constant amplitude cyclic histories are not sufficient for accurate determination of material behavior.

f) A small number of cycles or reversals are not sufficient to describe material response over a large number of cycles.

g) The parameter values selected from published data in handbooks and the literature are completely unreliable and the constitutive law computed from these parameters is inaccurate by 40% or more.

h) The kinematic law performs better than the isotropic law. The Chaboche and B-L laws have similar performance and are much better than the kinematic and isotropic laws.

i) The errors in stress (when the strain is given) with optimal parameters are of the order of 20% for the Chaboche and B-L laws and of the order of 30% for the kinematic and isotropic laws. The parameters for laws determined from published material property data in handbooks leads to errors of the order of 40%. The influence of the difference of the two materials is relatively small in comparison with inaccuracy of the constitutive law.

j) For problems with prescribed stress when strain is computed as the independent variable the constitutive laws perform extremely poorly with an error of more than 100%. In this case, better accuracy can be achieved when stress is the independent variable for determining parameters. This observation indicates that parameter determination has to be dependent on the goal of analysis.

k) Selecting the criterion for determining parameters from a single sample or from a set of samples is problematic.

l) The results presented are only for a one-dimensional case. For two- and three-dimensional cases, especially for nonproportional strains, the error is expected to be larger.

m) The data presented is for two particular aluminum alloys. The question exists as to whether the results have the same character for steel or other materials. No relevant data for comparison have been found in the literature.

n) There is a need for analyses of the type presented in this report to obtain reliable information about materials. Otherwise, errors of 30-40% or more in the constitutive law have to be assumed in computational analyses for engineering applications.

o) The reliability of any (e.g., FEM computation) numerical analysis of plasticity problems is questionable. Any usual deterministic formulation is of very problematic value. Very likely some approach based on Monte Carlo sensitivity analyses should be used which leads to brackets of desired information. Another possibility (still more desirable) is to develop a mathematical theory for direct computation of the brackets. The parameters and initial conditions of the internal variables have to be separately treated. The initial condition estimates related to the expected worst case past could be used. See [16] where this question was addressed in a simple setting.

Finally, let us remark on various aspects which were not analyzed directly in the paper. Some will be the topics of forthcoming reports.

a) Asymptotic behavior of stress for periodic strain and distinction between the transitional and stable parts of the stress response were not analyzed. The reason is that, in practice a purely periodic regime is usually unavailable and because the main emphasis of this paper is assessment of the constitutive law for the general case.

b) Corrections related to use of engineering stress and strain were not considered. For example, observed modulus of elasticity was not corrected by the effect of changing cross section under applied load. The reason is that this effect is not essential in our framework.

c) The effects of rate dependency, viscoelasticity, etc. were not considered. The reason is that the aluminum alloys were studied at room temperature and with a time scale such that rate dependent effects were small. These effects can be neglected in comparison with the other factors considered.

d) Because of limited funds samples were only oriented with the rolling direction of sheet fabrication, results were only for sheet material and experiments were only one dimensional.

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Appendix I

EXPERIMENTS AND MEASUREMENT OF DISPERSION

Background

A series of experiments was designed to obtain data about constitutive modeling uncertainty in a relatively simple setting. In particular, the experiments concentrate on uncertainties due to nonlinear material behavior. A fair comparison of various constitutive laws can be made when the parameters of such laws are determined with infinite precision. However, these parameters must be determined experimentally and the precision to which parameters can be determined is limited by the uncertainty associated with the experiments. An assessment of experimental uncertainty can be obtained by knowledge of the accuracy of instrumentation.

Instrumentation and Software

The tests were conducted in a MTS servo hydraulic testing machine with a 2000 pound full scale load cell, a 2 percent full scale extensometer, an IBM Personal Computer for data acquisition and control and a Hewlett Packard X-Y chart recorder to monitor progress of the experiment. Since loading was axial and reversed a reduced section specimen was used for the cyclic tests. A special collet adapter was designed to grip the specimen in the testing machine. ASTM-E606-80 provisions were followed for strain controlled cyclic fatigue tests.

The uncertainty in experimental data due to load and strain transducers is shown in Table I.1. The entries in Table I.1 are maximum or worst case values. For the specimen geometry used in this study uncertainty in terms of stress in the specimen is also shown in the table. Assuming the worst case and summing all uncertainties in the table would give 0.183 percent of full scale or 699 kPa (101 psi) as the uncertainty in stress. At full scale the load cell deflects approximately 0.0508 mm (0.002 inch). A similar analysis for strain is shown in Table I.2. Again assuming a worst case by summing all uncertainties would give 0.383 percent of full scale or 81.2 microstrain. The actuating force of the extensometer is approximately 60 grams and the weight is approximately 20 grams. These results include repeatable relative error, absolute error, and offset error.

Table I.1. Uncertainty associated with the load transducer.

Source	Percent of Full Scale	Specimen Stress kPa (psi)
Repeatability	0.03	115 (16.6)
Linearity	0.1	382 (55.4)
Hysteresis	0.05	191 (27.7)
Temperature effect on zero	0.001	3.82 (0.554)
Temperature effect on output	0.002	7.64 (1.11)

Table I.2. Uncertainty associated with the strain transducer.

Source	Percent of Full Scale	Specimen Strain (microstrain)
Repeatability	0.03	0.636
Linearity	0.2	4.24
Hysteresis	0.15	3.18
Temperature effect on zero	0.001	0.0212
Temperature effect on output	0.002	0.0424

Experiments were performed under strain control with the command signal generated by a computer. The command signal was generated with a 12 bit D/A converter so the desired command was programmed to 1 part in 2048 or 0.0488 percent of full scale. At predetermined time intervals the computer recorded load and strain values. Data was acquired by a 16 bit A/D converter so the resolution of recorded stress and strain data was 1 part in 32768 or 0.00305 percent of full scale. After each test was completed data was scaled to stress and strain and stored on floppy diskettes. The recorder was used to produce load-strain plots for different reversals to verify that tests were progressing properly. The control loop was tuned to give the best dynamic response possible. Since a ramp was used to program the strain history the discontinuous slope at reversal points resulted in a servo loop control error in strain of 0.2 percent of full scale. The laboratory temperature was controlled to within one degree centigrade. For a nominal modulus of 72 Gpa and thermal expansion coefficient of 22 microstrain per degree centigrade a one degree temperature change would result in a thermal strain of 22 microstrain which in strain control would produce an error in stress of 0.4 percent of full scale.

Variation in the extensometer gage length and specimen cross sectional area would introduce an absolute error that depends on strain level. This type of error would introduce nonlinearity but would not affect repeatability and accuracy. Other factors such as specimen bending related to alignment, initial buckling, variation in microstructure texture and orientation effects due to processing are difficult to quantify but contribute to uncertainty. Considering the worst case of all sources of error in combination, an

estimate of experimental uncertainty in strain is approximately 0.588 percent of full scale (125 microstrain) and an estimate of experimental uncertainty in stress is approximately 0.586 percent of full scale (2.24 MPa 325 psi). This includes the point of strain reversal and thermal excursions in the laboratory. At points in the strain history other than reversals and over short periods when thermal offsets can be neglected the error is less. With these assumptions the experimental uncertainty in strain is approximately 0.385 percent of full scale (82 microstrain) and an estimate of the uncertainty in stress is approximately 0.183 percent of full scale (699 kPa 101 psi). If only repeatability is considered, the uncertainty for strain is 0.03 percent of full scale (6.4 microstrain) and for stress is 0.03 percent of full scale (115 kPa 16.6 psi).

Experiments and data base

For the testing program three different strain histories were used: a triangle (or constant amplitude) history, and two different random histories. For the three strain histories seven experiments were conducted with each test having a constant mean strain level ranging from -0.6% to 0.6%. All the tests had a strain range of 1.0% and a constant strain rate of 0.001/second. The tests were performed at room temperature 22°C and at a relative humidity of approximately 50%. The data was collected for approximately one half of the fatigue life of each sample and the test was discontinued. Replicate tests were conducted for each combination of strain history, mean strain level, and material supplier. Thus a total of 84 data sets constituted the database used to determine parameters for each constitutive model.

Appendix II

DETERMINATION OF PARAMETERS FROM PUBLISHED DATA

The constitutive law parameters can be estimated from data published in the literature and in handbooks. Esimates of the parameters were made from available data according to the following procedure. The monotonic and cyclic properties of 5454-H32 aluminum from published data [7,9,10,20] resulted in the partial information of Table II.1.

Table II.1. Monotonic and cyclic propeties of 5454 aluminum.

5454-H32	Plate	Sheet	Plate and Sheet
0.2% σ_y	25.5 ksi	31.1 ksi	---
Modulus, E	10,100 ksi	10,400 ksi	---
Strength coefficient, K	34.5 ksi	---	---
Strain hardening exponent, n	0.041	---	---
Cyclic yield strength, σ'_y	---	---	36.0 ksi
Cyclic strength coefficient, K'	---	---	55.0 ksi
Cyclic strain hardening exponent, n'	---	---	0.068

The monotonic stress strain curve is represented by

$$\sigma = K\varepsilon_p^n$$

$$\sigma_t = \varepsilon_e + \varepsilon_p = \frac{\sigma}{E} + \left(\frac{\sigma}{K}\right)^{1/n}.$$

The cyclic stress strain curve is represented by

$$\sigma = K'\varepsilon_p^{n'}$$

$$\Delta\varepsilon = \frac{\Delta\sigma}{E} + 2\left(\frac{\Delta\sigma}{2K'}\right)^{1/n'}$$

To estimate the missing data for sheet material assume that the strain hardening exponent does not change much from sheet to plate and determine the monotonic strength coefficient to match the reported yield strength for sheet material. The estimates for 5454-H32 sheet (which were not found in the literature) are 40.1 ksi for the strength coefficient and 0.41 for the strain hardening exponent.

One estimate of the parameters for the kinematic and isotropic constitutive laws is to match the hysteresis loop tip at the point of maximum strain in the given history to the monotonic stress strain curve.

Table II.2. Kinematic and isotropic law parameter estimates from published monotonic stress-strain data.

Property	Value
E	10,400 ksi
E_p	104 ksi
σ_y	32 ksi

A second estimate which should have less error in the cyclic response can be obtained by matching the hysteresis loop tip at the point of maximum strain in the given history to the cyclic stress strain curve.

Table II.3. Kinematic and isotropic law parameter estimates from published cyclic stress-strain data.

Property	Value
E	10,400 ksi
E_p	104 ksi
σ_y	38.6 ksi

An estimate of the parameters of the Chaboche law can be determined in two stages. First fit the kinematic coefficients to match the first reversal of the strain history using monotonic stress strain properties to determine the elastic parameters, E and σ_y, and the kinematic parameters, C and a. Second, the cyclic stress strain properties are used to fit the isotropic parameters, Q and b, for the 500th cycle of a 1% plastic strain history.

Table II.4. Chaboche law parameter estimates from published monotonic and cyclic stress-strain data.

Property	Value
E	10,400 ksi
a	6 ksi
C	4,000
σ_y	27 ksi
Q	10 ksi
b	1

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

> To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.
>
> To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.
>
> To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.
>
> To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.
>
> To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

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